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ON *n*-ABSORBING IDEALS OF COMMUTATIVE RINGS

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Let R be a commutative ring with $1 \neq 0$ and n a positive integer. In this article, we study two generalizations of a prime ideal. A proper ideal I of R is called an n-absorbing (resp., strongly n-absorbing) ideal if whenever $x_1 \cdots x_{n+1} \in I$ for $x_1, \ldots, x_{n+1} \in R$ (resp., $I_1 \cdots I_{n+1} \subseteq I$ for ideals $I_1, \ldots, I_{n+1}$ of R), then there are n of the x_i's (resp., n of the I_i's) whose product is in I. We investigate n-absorbing and strongly n-absorbing ideals, and we conjecture that these two concepts are equivalent. In particular, we study the stability of n-absorbing ideals with respect to various ring-theoretic constructions and study n-absorbing ideals in several classes of commutative rings. For example, in a Noetherian ring every proper ideal is an n-absorbing ideal for some positive integer n, and in a Prüfer domain, an ideal is an n-absorbing ideal for some positive integer n if and only if it is a product of prime ideals.

Key Words: 2-Absorbing ideal; *n*-Absorbing ideal; Prime; Prufer; Strongly *n*-absorbing ideal.

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1. INTRODUCTION

In this article, we study n-absorbing ideals in commutative rings with identity, which are a generalization of prime ideals. The concept of 2-absorbing ideals was introduced and investigated in [3]. Let n be a positive integer. A proper ideal I of a commutative ring R is called an *n-absorbing ideal* if whenever $x_1 \cdots x_{n+1} \in I$ for $x_1, \ldots, x_{n+1} \in R$, then there are n of the x_i's whose product is in I. Equivalently, a proper ideal I of R is an n-absorbing ideal if and only if whenever $x_1 \cdots x_m \in I$ for $x_1, \ldots, x_m \in R$ with $m > n$, then there are n of the x_i's whose product is in I. In terms of factor rings, I is an n-absorbing ideal of R if and only if whenever the product of $n+1$ elements of R/I is 0, then the product of some n of these elements is 0 in R/I. Thus a 1-absorbing ideal is just a prime ideal. More generally, we show that the intersection of n prime ideals, the product of n maximal ideals, the nth symbolic power of a prime ideal, the product of n principal prime ideals

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in an integral domain, and (divisorial) ideals which are the v-product of n height-one prime ideals in a Krull domain are all n-absorbing ideals. For principal ideals in an integral domain, this concept has been studied with respect to nonunique factorization in [2]. Other generalizations of prime ideals have recently been studied in [1].

In Section 2, we give some basic properties of n-absorbing ideals. For example, we show that an n-absorbing ideal has at most n minimal prime ideals (Theorem 2.5), that the product of n maximal ideals is an n-absorbing ideal (Theorem 2.9), and that if an n-absorbing ideal I has exactly n minimal prime ideals $P_1, \ldots, P_n$, then $P_1 \cdots P_n \subseteq I$ (Theorem 2.14). However, the product of n prime ideals need not be an n-absorbing ideal (Example 2.7). Section 3 continues the study of basic properties of n-absorbing ideals. In particular, we discuss the relationship between primary ideals and n-absorbing ideals and investigate when $(I :_R x)$ is an n-absorbing ideal of R for I a proper ideal of R.

In Section 4, we study the stability of n-absorbing ideals with respect to various ring-theoretic constructions such as localization, factor rings, and idealization. In particular, we determine the n-absorbing ideals in the direct product of a finite number of rings (Corollary 4.8) and in integral domains of the form $D + XK[[X]]$, where D is a subring of a field K (Theorem 4.17). In Section 5, we study n-absorbing ideals in several classes of commutative rings. For example, we show that every proper ideal of a Noetherian ring is an n-absorbing ideal for some positive integer n (Theorem 5.3) and that an ideal I of a valuation domain R is an n-absorbing ideal of R if and only if $I = P^m$, where $P = Rad(I)$ is a prime ideal of R and $1 \leq m \leq n$ (Theorem 5.5). More generally, an ideal of a Prüfer domain is an n-absorbing ideal for some positive integer n if and only if it is a product of prime ideals (Theorem 5.7). We also discuss for which positive integers n, a ring R has an ideal which is n-absorbing, but not $(n-1)$-absorbing.

In the final section, we study another generalization of prime ideal. We define a proper ideal I of a ring R to be a *strongly n-absorbing ideal* if whenever $I_1 \cdots I_{n+1} \subseteq I$ for ideals $I_1, \ldots, I_{n+1}$ of R, then the product of some n of the I_j's is contained in I. Thus a strongly 1-absorbing ideal is just a prime ideal. Clearly a strongly n-absorbing ideal of R is also an n-absorbing ideal of R, and we conjecture that these two concepts are equivalent (we show they are equivalent for Prüfer domains in Corollary 6.9). We also give several results relating strongly n-absorbing ideals to earlier material. For example, we show that if I is a strongly n-absorbing ideal with $m(\leq n)$ minimal prime ideals $P_1, \ldots, P_m$, then ${P_1}^{n_1} \cdots {P_m}^{n_m} \subseteq I$ for positive integers $n_1, \ldots, n_m$ with $n = n_1 + \cdots + n_m$ (Theorem 6.2), that the product of n maximal ideals is a strongly n-absorbing ideal (Corollary 6.7), and that every proper ideal of a Noetherian ring is a strongly n-absorbing ideal for some positive integer n (Corollary 6.8).

As mentioned above, the concept of 2-absorbing ideals was introduced and studied in [3]. In some cases, results about 2-absorbing ideals generalize in the natural way to n-absorbing ideals for $n \geq 3$; in other cases they do not (see Example 4.11(c) for instance). And in a few cases, we have been unable to determine if results extend or not (see Theorem 4.15 and Section 6).

We assume throughout that all rings are commutative with $1 \neq 0$ and that $f(1) = 1$ for all ring homomorphisms $f : R \longrightarrow T$. Let R be a ring. Then $\dim(R)$ denotes the Krull dimension of R, $Spec(R)$ denotes the set of prime ideals of R,

$Max(R)$ denotes the set of maximal ideals of R, $T(R)$ denotes the total quotient ring of R, $qf(R)$ denotes the quotient field of R when R is an integral domain, $Nil(R)$ denotes the ideal of nilpotent elements of R, and $Z(R)$ denotes the set of zero-divisors of R. If I is a proper ideal of R, then $Rad(I)$ and $Min_R(I)$ denote the radical ideal of I and the set of prime ideals of R minimal over I, respectively. We will often let 0 denote the zero ideal.

We start by recalling some background material. A prime ideal P of a ring R is said to be a *divided prime ideal* if $P \subset xR$ for every $x \in R \setminus P$; thus a divided prime ideal is comparable to every ideal of R. An integral domain R is said to be a *divided domain* if every prime ideal of R is a divided prime ideal.

An integral domain R is said to be a *valuation domain* if either $x \mid y$ or $y \mid x$ (in R) for all $0 \neq x, y \in R$ (a valuation domain is a divided domain). If I is a nonzero fractional ideal of a ring R, then $I^{-1} = \{x \in T(R) \mid xI \subseteq R\}$. An integral domain R is called a *Dedekind* (resp., *Prüfer*) *domain* if $II^{-1} = R$ for every nonzero fractional ideal (resp., finitely generated fractional ideal) I of R. Moreover, an integral domain R is a Prüfer domain if and only if R_M is a valuation domain for every maximal ideal M of R. An integral domain R is called an *almost Dedekind domain* if R_M is a Noetherian valuation domain (DVR) for every maximal ideal M of R. An almost Dedekind domain is a Prüfer domain with $\dim(R) \leq 1$. A ring R is a *Bézout ring* if every finitely generated ideal of R is principal. As usual, for a nonzero fractional ideal I of an integral domain R, we define $I_v = (I^{-1})^{-1}$ and say that I is *divisorial* (or a *v-ideal*) if $I_v = I$.

Several of our examples use the $R(+)M$ construction. Let R be a ring and M an R-module. Then $R(+)M = R \times M$ is a ring with identity $(1, 0)$ under addition defined by $(r, m) + (s, n) = (r + s, m + n)$ and multiplication defined by $(r, m)(s, n) = (rs, rn + sm)$. Note that $(0(+)M)^2 = 0$; so $0(+)M \subseteq Nil(R(+)M)$. We view R as a subring of $R(+)M$ via $r \mapsto (r, 0)$.

As usual, $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Z}_n$, $\mathbb{Q}$, and $\mathbb{R}$ will denote the positive integers, integers, integers modulo n, rational numbers, and real numbers, respectively. We define $n + \infty = \infty + \infty = \infty$ for all $n \in \mathbb{Z}$. We will use "$\subset$" to denote proper inclusion. For any undefined concepts or terminology, see [6, 8, 9], or [10].

2. BASIC PROPERTIES OF *n*-ABSORBING IDEALS

Let n be a positive integer. Recall that a proper ideal I of a ring R is an n-absorbing ideal of R if whenever $x_1 \cdots x_{n+1} \in I$ for $x_1, \ldots, x_{n+1} \in R$, then there are n of the x_i's whose product is in I. In this section, we give some basic properties of n-absorbing ideals. We start with several elementary results.

Theorem 2.1. *Let R be a ring, and let m and n be positive integers.*

(a) *A proper ideal I of R is an n-absorbing ideal if and only if whenever $x_1 \cdots x_m \in I$ for $x_1, \ldots, x_m \in R$ with $m > n$, then there are n of the x_i's whose product is in I.*
(b) *If I is an n-absorbing ideal of R, then I is an m-absorbing ideal of R for all $m \geq n$.*
(c) *If I_j is an n_j-absorbing ideal of R for each $1 \leq j \leq m$, then $I_1 \cap \cdots \cap I_m$ is an n-absorbing ideal of R for $n = n_1 + \cdots + n_m$. In particular, if $P_1, \ldots, P_n$ are prime ideals of R, then $P_1 \cap \cdots \cap P_n$ is an n-absorbing ideal of R.*

(d) *If $p_1, \ldots, p_n$ are prime elements of an integral domain R, then $I = p_1 \cdots p_n R$ is an n-absorbing ideal of R.*

(e) *If I is an n-absorbing ideal of R, then $Rad(I)$ is an n-absorbing ideal of R and $x^n \in I$ for all $x \in Rad(I)$.*

Proof. The proofs of (a), (b), (c), and (d) are all routine, and thus they are omitted.

(e) Let I be an n-absorbing ideal of R. Hence $x^n \in I$ for all $x \in Rad(I)$. Let $x_1 \cdots x_{n+1} \in Rad(I)$ for $x_1, \ldots, x_{n+1} \in R$. Then $x_1^n \cdots x_{n+1}^n = (x_1 \cdots x_{n+1})^n \in I$. Since I is an n-absorbing ideal, we may assume that $x_1^n \cdots x_n^n \in I$. Thus $(x_1 \cdots x_n)^n = x_1^n \cdots x_n^n \in I$, and hence $x_1 \cdots x_n \in Rad(I)$. Thus $Rad(I)$ is an n-absorbing ideal of R. □

Let I be a proper ideal of a ring R. In Theorem 2.1(b), we observed that an n-absorbing ideal is also an m-absorbing ideal for all integers $m \geq n$. If I is an n-absorbing ideal of R for some positive integer n, then define $\omega_R(I) = \min\{n \mid I \text{ is an } n\text{-absorbing ideal of } R\}$; otherwise, set $\omega_R(I) = \infty$ (we will just write $\omega(I)$ when the context is clear). It is convenient to define $\omega(R) = 0$. Thus for any ideal I of R, we have $\omega(I) \in \mathbb{N} \cup \{0, \infty\}$ with $\omega(I) = 1$ if and only if I is a prime ideal of R and $\omega(I) = 0$ if and only if $I = R$. So $\omega(I)$ measures, in some sense, how far I is from being a prime ideal of R. When R is an integral domain and $0 \neq x \in R$, we have $\omega(xR) = \omega(x)$ as defined in [2, 7].

Remark 2.2. Several of the results in Theorem 2.1 may be recast using the ω function. For example, Theorem 2.1(c) becomes $\omega(I_1 \cap \cdots \cap I_m) \leq \omega(I_1) + \cdots + \omega(I_m)$. In particular, $\omega(P_1 \cap \cdots \cap P_n) \leq n$ when $P_1, \ldots, P_n$ are prime ideals of R. Easy examples show that both inequalities may be strict. However, if $P_1, \ldots, P_n$ are incomparable prime ideals of R, then $\omega(P_1 \cap \cdots \cap P_n) = n$. (Choose $x_i \in P_i \setminus \bigcup_{j \neq i} P_j$ for each $1 \leq i \leq n$. Then $x_1 \cdots x_n \in P_1 \cap \cdots \cap P_n$, but no proper subproduct of the x_i's is in $P_1 \cap \cdots \cap P_n$. Thus $\omega(P_1 \cap \cdots \cap P_n) \geq n$.) Theorem 2.1(d) becomes $\omega(p_1 \cdots p_n R) = n$, where $p_1, \ldots, p_n$ are prime elements of an integral domain R. More generally, $\omega(x_1 \cdots x_n R) \geq n$ for any nonzero, nonunits x_i in an integral domain R. Also, Theorem 2.1(e) may be restated as $\omega(Rad(I)) \leq \omega(I)$. Again, easy examples show that both inequalities may be strict.

We next give a very elementary example of a ring with proper ideals which are not n-absorbing for any positive integer n. For other examples, see Examples 4.12, 4.18, and 5.6.

Example 2.3. Let $R = \prod_{i=1}^{\infty} \mathbb{Z}_2$. Then R is a von Neumann regular ring (i.e., R is reduced with $\dim(R) = 0$). Let $I_n = \{(x_i) \in R \mid x_i = 0 \text{ for } 1 \leq i \leq n\}$ for each positive integer n, and let $I = \{(x_i) \in R \mid x_{2i-1} = 0 \text{ for all } i \in \mathbb{N}\}$. Then it is easily verified that I_n and I are proper ideals of R with $\omega(I_n) = n$ for each positive integer n and $\omega(I) = \infty$. Note that each I_n is the product of n maximal ideals of R. It is also easily verified that $\omega(0) = \infty$.

The first major result of this section (Theorem 2.5) is that an n-absorbing ideal has at most n minimal prime ideals. We will need the following lemma.

Lemma 2.4 ([9, Theorem 2.1, p. 2]). *Let $I \subseteq P$ be ideals of a ring R with P a prime ideal. Then the following statements are equivalent*:

(1) *P is a minimal prime ideal of I;*
(2) *For each $x \in P$, there is a $y \in R \backslash P$ and a positive integer n such that $yx^n \in I$.*

Theorem 2.5. *Let I be an n-absorbing ideal of a ring R. Then there are at most n prime ideals of R minimal over I. Moreover, $|Min_R(I)| \leq \omega_R(I)$.*

Proof. We may assume that $n \geq 2$ since a 1-absorbing ideal is a prime ideal. Suppose that $P_1, \ldots, P_{n+1}$ are distinct prime ideals of R minimal over I. Thus for each $1 \leq i \leq n$, there is an $x_i \in P_i \backslash ((\bigcup_{k \neq i} P_k) \cup P_{n+1})$. By Lemma 2.4, for each $1 \leq i \leq n$, there is a $c_i \in R \backslash P_i$ such that $c_i x_i^{n_i} \in I$ for some integer $n_i \geq 1$. Since $I \subseteq P_{n+1}$ is an n-absorbing ideal of R and $x_i \notin P_{n+1}$ for each $1 \leq i \leq n$, we have $c_i x_i^{n-1} \in I$ for each $1 \leq i \leq n$, and hence $(c_1 + \cdots + c_n) x_1^{n-1} \cdots x_n^{n-1} \in I$. Since $x_i \in P_i \backslash (\bigcup_{k \neq i} P_k)$ and $c_i x_i^{n-1} \in I \subseteq P_1 \cap \cdots \cap P_n$ for each $1 \leq i \leq n$, we have $c_i \in (\bigcap_{k \neq i} P_k) \backslash P_i$ for each $1 \leq i \leq n$, and thus $c_1 + \cdots + c_n \notin P_i$ for each $1 \leq i \leq n$. Hence $(c_1 + \cdots + c_n) \prod_{k \neq i} x_k^{n-1} \notin P_i$ for each $1 \leq i \leq n$; so $(c_1 + \cdots + c_n) \prod_{k \neq i} x_k^{n-1} \notin I$ for each $1 \leq i \leq n$, and thus $x_1^{n-1} \cdots x_n^{n-1} \in I \subseteq P_{n+1}$ since I is an n-absorbing ideal of R. But then $x_i \in P_{n+1}$ for some $1 \leq i \leq n$, which is a contradiction. Hence there are at most n prime ideals of R minimal over I.

The "moreover" statement is clear. □

Let $n \geq m$ be positive integers. Then there is an n-absorbing, but not $(n-1)$-absorbing, ideal of a ring R that has exactly m minimal prime ideals. For example, let $n = 3$. Then the ideals $I_1 = 27\mathbb{Z}$, $I_2 = 18\mathbb{Z}$, and $I_3 = 30\mathbb{Z}$ are 3-absorbing, but not 2-absorbing, ideals of $\mathbb{Z}$ with one, two, and three minimal prime ideals, respectively. More generally, let $p_1, \ldots, p_m \in \mathbb{Z}$ be distinct positive primes and $n_1, \ldots, n_m$ be positive integers with $n = n_1 + \cdots + n_m$. Then $I = p_1^{n_1} \cdots p_m^{n_m}\mathbb{Z}$ is an n-absorbing, but not $(n-1)$-absorbing, ideal of $\mathbb{Z}$ with exactly m minimal prime ideals, namely, $p_1\mathbb{Z}, \ldots, p_m\mathbb{Z}$, i.e., $|Min_{\mathbb{Z}}(I)| = m$ and $\omega_{\mathbb{Z}}(I) = n$ (cf. Theorems 2.1(d) and 2.9).

We have observed in Theorem 2.1(c) that the intersection of n prime ideals of a ring R is always an n-absorbing ideal of R. We next investigate when the product of n prime ideals of R is an n-absorbing ideal of R. Note that if $P_1, \ldots, P_n$ are incomparable prime ideals of R, then $\omega(P_1 \cdots P_n) \geq n$. (Let $x_i \in P_i \backslash (\bigcup_{j \neq i} P_j)$ for each $1 \leq i \leq n$. Then $x_1 \cdots x_n \in P_1 \cdots P_n$, but no proper subproduct of the x_i's is in $P_1 \cdots P_n$.) Also, the proof of Lemma 2.8 shows that $\omega(P^n) \geq n$ for P a prime ideal of R with $P^{n+1} \subset P^n$ (cf. Theorem 6.3). It has already been noted in Theorem 2.1(d) that the product of n nonzero principal prime ideals in an integral domain R is an n-absorbing ideal of R. The next theorem gives another trivial case where the product of n prime ideals of R is an n-absorbing ideal of R (see Corollary 4.9 for a generalization).

Theorem 2.6. *Let $P_1, \ldots, P_n$ be prime ideals of a ring R that are pairwise comaximal. Then $I = P_1 \cdots P_n$ is an n-absorbing ideal of R. Moreover, $\omega(I) = n$.*

Proof. Since the P_i's are pairwise comaximal, we have $I = P_1 \cdots P_n = P_1 \cap \cdots \cap P_n$. Thus I is an n-absorbing ideal of R by Theorem 2.1(c).

The "moreover" statement follows from comments in Remark 2.2 since $P_1, \ldots, P_n$ are incomparable. □

In general, the product of $n \geq 2$ prime ideals of a ring R need not be an n-absorbing ideal of R. We have the following examples. However, see Corollary 4.4.

Example 2.7. (a) Let $R = \mathbb{Z}[X, Y] + 6Z\mathbb{Z}[X, Y, Z] \subset \mathbb{Z}[X, Y, Z]$. Then $P_1 = X\mathbb{Z}[X, Y] + 6Z\mathbb{Z}[X, Y, Z]$ and $P_2 = Y\mathbb{Z}[X, Y] + 6Z\mathbb{Z}[X, Y, Z]$ are incomparable prime ideals of R. However, $I = P_1P_2$ is not a 2-absorbing ideal of R since $2 \cdot 3 \cdot 6Z^2 \in I$, but $2 \cdot 3 \notin I$, $2 \cdot 6Z^2 \notin I$, and $3 \cdot 6Z^2 \notin I$. Similarly, P_1^2 and P_2^2 are not 2-absorbing ideals of R.

(b) Let $R = \mathbb{Z}[X, Y, Z]$. Then $P_1 = (2, X), P_2 = (2, Y)$, and $P_3 = (2, Z)$ are incomparable (nonmaximal) prime ideals of R. However, $I = P_1P_2P_3 = (8, 4X, 4Y, 4Z, 2XY, 2XZ, 2YZ, XYZ)$ is not a 3-absorbing ideal of R. To see this, let $f_1 = 2, f_2 = X + Y + 2, f_3 = X + Z + 2$, and $f_4 = Y + Z + 2$. Then $f_1f_2f_3f_4 \in I$, but no product of any 3 of the f_i's is in I. (Note that every ideal of R is an n-absorbing ideal for some positive integer n by Theorem 5.3.)

(c) We next generalize part (a). Let m and n be integers with $2 \leq n \leq m$, and let $p_1, \ldots, p_m \in \mathbb{Z}$ be the first m positive primes. For $q_m = p_1 \cdots p_m$, let $R = \mathbb{Z}[X_1, \ldots, X_n] + q_mY\mathbb{Z}[X_1, \ldots, X_n, Y]$, a subring of $\mathbb{Z}[X_1, \ldots, X_n, Y]$, and let $P_i = X_i\mathbb{Z}[X_1, \ldots, X_n] + q_mY\mathbb{Z}[X_1, \ldots, X_n, Y]$ for each $1 \leq i \leq n$. Then $P_1, \ldots, P_n$ are incomparable prime ideals of R. However, $I = P_1 \cdots P_n$ is not an m-absorbing ideal of R (and hence I is also not an n-absorbing ideal of R). To see this, let $q_m^* = q_m^{n-1}Y^n$. Then $p_1 \cdots p_mq_m^* = (q_mY)^n \in I$, but no proper subproduct is in I. Similarly, let $J = P_1^{n_1} \cdots P_n^{n_n}$ for integers $n_k \geq 0$ with $n = n_1 + \cdots + n_n$; then J is not an m-absorbing ideal of R.

(d) More generally, one can ask how $\omega(IJ), \omega(I)$, and $\omega(J)$ compare when I and J are proper ideals of a ring R. If I and J are comaximal, then $\omega(IJ) = \omega(I) + \omega(J)$ by Corollary 4.9. However, the two examples above show that we may have $\omega(IJ) > \omega(I) + \omega(J)$ even when I and J are prime ideals of R. We may also have $\omega(IJ) < \omega(I) + \omega(J)$. This is trivially true if $I = J$ is an idempotent prime ideal of R. For a less trivial example, let $P \subset Q$ be nonzero prime ideals of a valuation domain R. Then it is easy to verify that $PQ = P$, and thus $\omega(PQ) = 1 < 2 = \omega(P) + \omega(Q)$. We have already observed that for incomparable prime ideals P and Q of R, we have $\omega(PQ) \geq 2 = \omega(P) + \omega(Q)$. Also, for any integral domain R, we have $\omega(xyR) \leq \omega(xR) + \omega(yR)$ for all $0 \neq x, y \in R$ by [2, Theorem 2.3].

If $M_1, \ldots, M_n$ are distinct maximal ideals of a ring R, then $I = M_1 \cdots M_n$ is an n-absorbing ideal of R by Theorem 2.6. We next show that the product of any n maximal ideals of R is an n-absorbing ideal of R, but first we show that M^n is an n-absorbing ideal of R for any maximal ideal M of R (cf. Theorem 3.1). Note that we may have $\omega(M^n) < n$. For example, this would happen if $M^k = M^{k+1}$ for some integer k with $1 \leq k \leq n - 1$ (cf. Remark 6.4(a)).

Lemma 2.8. *Let M be a maximal ideal of a ring R and n a positive integer. Then M^n is an n-absorbing ideal of R. Moreover, $\omega(M^n) \leq n$, and $\omega(M^n) = n$ if $M^{n+1} \subset M^n$.*

Proof. Let $x_1 \cdots x_{n+1} \in M^n$ for $x_1, \ldots, x_{n+1} \in R$. If $x_1, \ldots, x_{n+1} \in M$, then we are done; so we may assume that $x_{n+1} \notin M$. Then $(M^n, \ x_{n+1}) = R$; so $y + x_{n+1}z = 1$ for some $y \in M^n$ and $z \in R$. Thus $x_1 \cdots x_n = (x_1 \cdots x_n)1 = (x_1 \cdots x_n)y + (x_1 \cdots x_{n+1})z \in M^n$, and hence M^n is an n-absorbing ideal of R.

The first part of the "moreover" statement is clear. Now suppose that $M^{n+1} \subset M^n$. Then there are $x_1, \ldots, x_n \in M$ such that $x_1 \cdots x_n \in M^n \backslash M^{n+1}$. Thus no product of $n-1$ of the x_i's is in M^n since otherwise $x_1 \cdots x_n \in M^{n+1}$, a contradiction. Hence M^n is not an $(n-1)$-absorbing ideal of R, and thus $\omega(M^n) = n$ since we showed above that M^n is an n-absorbing ideal of R. □

Theorem 2.9. *Let $M_1, \ldots, M_n$ be maximal ideals of a ring R. Then $I = M_1 \cdots M_n$ is an n-absorbing ideal of R. Moreover, $\omega(I) \leq n$.*

Proof. We show that if $M_1, \ldots, M_m$ are distinct maximal ideals of R and $n_1, \ldots, n_m$ are positive integers with $n = n_1 + \cdots + n_m$, then $I = M_1^{n_1} \cdots M_m^{n_m}$ is an n-absorbing ideal of R. By Lemma 2.8, each $M_i^{n_i}$ is a n_i-absorbing ideal of R. Thus $I = M_1^{n_1} \cdots M_m^{n_m} = M_1^{n_1} \cap \cdots \cap M_m^{n_m}$ is an n-absorbing ideal of R by Theorem 2.1(c).

The "moreover" statement is clear. □

Our next goal (Theorem 2.14) is to show that if an n-absorbing ideal I has exactly n minimal prime ideals, say $P_1, \ldots, P_n$, then $P_1 \cdots P_n \subseteq I$ $(\subseteq P_1 \cap \cdots \cap P_n)$. Note that an n-absorbing ideal I of R has exactly n minimal prime ideals if and only if $|Min_R(I)| = \omega_R(I) = n$ by Theorem 2.5. First an example and two lemmas.

Example 2.10. Let $P_1, \ldots, P_n$ be incomparable prime ideals of a ring R, and let $I = P_1 \cap \cdots \cap P_n$. Then $Rad(I) = I, \omega(I) = n$, and $P_1 \cdots P_n \subseteq I = P_1 \cap \cdots \cap P_n$. However, the inclusion may be strict. For example, let $R = \mathbb{Z}[X, Y]$, $P_1 = (2, X)$, and $P_2 = (2, Y)$. Then $(4, 2X, 2Y, XY) = P_1P_2 \subset I = P_1 \cap P_2$ since $2 \in I \backslash P_1P_2$.

Lemma 2.11. *Let $n \geq 2$ and $P_1, \ldots, P_n$ be incomparable primes ideals of a ring R, and let I be an n-absorbing ideal of R contained in $P_1 \cap \cdots \cap P_n$. If $x_1^{m_1} \cdots x_n^{m_n} \in I$ for positive integers m_i and $x_i \in P_i \backslash (\bigcup_{k \neq i} P_k)$, then $x_1 \cdots x_n \in I$.*

Proof. Since I is an n-absorbing ideal of R, we have $x_1^{k_1} \cdots x_n^{k_n} \in I$ for integers $k_1, \ldots, k_n$ with each $0 \leq k_i \leq m_i$ and $k_1 + \cdots + k_n = n$. If some $k_i = 0$, say $k_1 = 0$, then $x_2^{k_2} \cdots x_n^{k_n} \in I \subseteq P_1$, a contradiction since $x_i \notin P_1$ for each $2 \leq i \leq n$. Thus $x_1 \cdots x_n \in I$. □

In the following results, we use the notation $P_j \prod_{i \neq j} c_i$ to represent the set of all products of the form $a \prod_{i \neq j} c_i$, where $a \in P_j$.

Lemma 2.12. *Let $n \geq 2$ and I be an n-absorbing ideal of a ring R such that I has exactly n minimal prime ideals, say $P_1, \ldots, P_n$. Let $1 \leq j \leq n$, and for every $i \neq j$ with $1 \leq i \leq n$, let $c_i \in P_i \backslash (\bigcup_{k \neq i} P_k)$. Then $P_j \prod_{i \neq j} c_i \in I$.*

Proof. Let $a \in P_j$. If $a \in P_j \backslash (\bigcup_{i \neq j} P_i)$, then $a \prod_{i \neq j} c_i \in I$ by Theorem 2.1(e) and Lemma 2.11. Now suppose that $a \in P_j \cap (\bigcup_{i \neq j} P_i)$. Let $d \in P_j \backslash (\bigcup_{i \neq j} P_i)$. We will find an element $b \in R$ such that $bd + a \in P_j \backslash (\bigcup_{i \neq j} P_i)$. Let $F = \{m \mid a \notin P_m$ for

$1 \leq m \leq n\}$, $D = \{m \mid a \in P_m$ for $1 \leq m \leq n, m \neq j\}$, $b = \prod_{k\in F} c_k$ (let $b = 1$ if $F = \emptyset$), and $x = bd + a$. Since $d\prod_{k\in F} c_k \in P_m$ and $a \notin P_m$ for every $m \in F$, we have $x \notin P_m$ for every $m \in F$. Since $a \in P_m$ for every $m \in D$ and $d\prod_{k\in F} c_k \notin P_m$ for every $m \in D$, we have $x \notin P_m$ for every $m \in D$. Thus $x \in P_j\backslash(\bigcup_{i\neq j} P_i)$, and hence $x\prod_{i\neq j} c_i \in I$ and $d\prod_{i\neq j} c_i \in I$ as above. Thus $(\prod_{k\in F} c_k)(d\prod_{i\neq j} c_i) + a\prod_{i\neq j} c_i = x\prod_{i\neq j} c_i \in I$, and hence $a\prod_{i\neq j} c_i \in I$. Thus $P_j\prod_{i\neq j} c_i \in I$. □

In view of the proof of Lemma 2.12, we have the following corollary.

Corollary 2.13. *Let $n \geq 2$ and $P_1, \ldots, P_n$ be incomparable prime ideals of a ring R. Let $a \in P_j$ for some $1 \leq j \leq n$. Then there is an element $d \in P_j\backslash(\bigcup_{i\neq j} P_i)$ and $b \in R$ such that $bd + a \in P_j\backslash(\bigcup_{i\neq j} P_i)$.*

We are now ready for the main result of this section. Example 2.10 shows that the inclusion in Theorem 2.14 may be proper, while Corollary 2.15 gives several cases where equality holds.

Theorem 2.14. *Let I be an n-absorbing ideal of a ring R such that I has exactly n minimal prime ideals, say $P_1, \ldots, P_n$. Then $P_1 \cdots P_n \subseteq I$. Moreover, $\omega(I) = n$.*

Proof. We may assume that $n \geq 2$ since a 1-absorbing ideal is a prime ideal. Let $a_i \in P_i$ for each $1 \leq i \leq n$. Then $a_1\prod_{2\leq i\leq n} c_i \in I$ for any choices $c_i \in P_i\backslash(P_1 \cup (\bigcup_{j\neq i} P_j))$, $2 \leq i \leq n$, by Lemma 2.12. Now suppose that for some $1 \leq k \leq n-1$, we have that $(a_1 \cdots a_k)\prod_{(k+1)\leq i\leq n} c_i \in I$ for any choices $c_i \in P_i\backslash(P_1 \cup (\bigcup_{j\neq i} P_j))$, $k+1 \leq i \leq n$; we will show that $(a_1 \cdots a_{k+1})\prod_{(k+2)\leq i\leq n} c_i \in I$ for any choices $c_i \in P_i\backslash(P_1 \cup (\bigcup_{j\neq i} P_j))$, $k+2 \leq i \leq n$. By Corollary 2.13, there is a $d_{k+1} \in P_{k+1}\backslash(\bigcup_{j\neq k+1} P_j)$ and $b_{k+1} \in R$ such that $b_{k+1}d_{k+1} + a_{k+1} \in P_{k+1}\backslash(\bigcup_{j\neq k+1} P_j)$. Put $c_{k+1} = b_{k+1}d_{k+1} + a_{k+1}$. Then by assumption, we have $((b_{k+1}a_1 \cdots a_k d_{k+1})\prod_{(k+2)\leq i\leq n} c_i) + ((a_1 \cdots a_{k+1})\prod_{(k+2)\leq i\leq n} c_i) = (a_1 \cdots a_k)(b_{k+1}d_{k+1} + a_{k+1})\prod_{(k+2)\leq i\leq n} c_i = (a_1 \cdots a_k)\prod_{(k+1)\leq i\leq n} c_i \in I$. Since $d_{k+1} \in P_{k+1}\backslash(\bigcup_{i\neq k+1} P_i)$, we have $(b_{k+1}a_1 \cdots a_k d_{k+1})\prod_{(k+2)\leq i\leq n} c_i \in I$ by assumption, and hence $(a_1 \cdots a_{k+1})\prod_{(k+2)\leq i\leq n} c_i \in I$. In particular, if $k = n-1$, then $(a_1 \cdots a_{n-1})(b_n d_n + a_n) \in I$, and thus $a_1 \cdots a_n \in I$. Hence $P_1 \cdots P_n \subseteq I$.

For the "moreover" statement, we have $\omega(I) \leq n$ since I is an n-absorbing ideal of R. For the reverse inequality, choose $x_i \in P_i\backslash(\bigcup_{j\neq i} P_j)$ for each $1 \leq i \leq n$. Then $x_1 \cdots x_n \in P_1 \cdots P_n \subseteq I$ by above. However, if some proper subproduct of the x_i's is in I, say $x_2 \cdots x_n \in I \subseteq P_1$, then $x_i \in P_1$ for some $2 \leq i \leq n$, a contradiction. Thus $\omega(I) = n$. □

Corollary 2.15. *Let I be an n-absorbing ideal of a ring R such that I has exactly n minimal prime ideals, say $P_1, \ldots, P_n$. If the P_i's are comaximal, then $I = P_1 \cdots P_n$. Moreover, $\omega(I) = n$. In particular, this holds if either each P_i is maximal,* $\dim(R) = 0$, *or R is an integral domain with* $\dim(R) \leq 1$.

Proof. We have $P_1 \cdots P_n \subseteq I \subseteq P_1 \cap \cdots \cap P_n$ by Theorem 2.14 and $P_1 \cap \cdots \cap P_n = P_1 \cdots P_n$ since the P_i's are comaximal. Thus $I = P_1 \cdots P_n$.

The "moreover" and "in particular" statements are clear. □

Corollary 2.16. *Let I be an n-absorbing ideal of a ring R such that I has exactly n minimal prime ideals, say $P_1, \ldots, P_n$. Then $I_{P_i} = P_{i_{P_i}}$ (in R_{P_i}) for all $1 \leq i \leq n$.*

Proof. If $n = 1$, then I is a prime ideal; so we may assume that $n \geq 2$. Let $1 \leq i \leq n$. Clearly, $I_{P_i} \subseteq P_{i_{P_i}}$ (in R_{P_i}). For the reverse inclusion, let $x \in P_i$. For every $1 \leq j \leq n$ such that $j \neq i$, let $c_j \in P_j \setminus (\bigcup_{k \neq j} P_k)$; then $c = \prod_{j \neq i} c_j \in R \setminus P_i$. Since $P_1 \cdots P_n \subseteq I$ by Theorem 2.14, we have $cx \in I$. Thus $x/s = cx/cs \in I_{P_i}$ for all $s \in R \setminus P_i$, and hence $I_{P_i} = P_{i_{P_i}}$. (For an alternate proof, just localize the inclusion $P_1 \cdots P_n \subseteq I \subseteq P_i$ at P_i.) □

In Section 6, we consider the case when $|Min_R(I)| < \omega_R(I)$, and we conjecture that if I is an n-absorbing ideal of a ring R such that I has exactly m minimal prime ideals $P_1, \ldots, P_m$ ($m \leq n$ by Theorem 2.5), then ${P_1}^{n_1} \cdots {P_m}^{n_m} \subseteq I$ for positive integers $n_1, \ldots, n_m$ with $n = n_1 + \cdots + n_m$ (see Theorem 6.2).

3. BASIC PROPERTIES OF *n*-ABSORBING IDEALS, II

In this section, we continue the study of basic properties of n-absorbing ideals begun in the previous section. We first consider the relationship between n-absorbing ideals and primary ideals. Our next result is a generalization of Lemma 2.8 since any power of a maximal ideal M is M-primary (also see Theorem 6.3, Remark 6.4(a), and Theorem 6.6).

Theorem 3.1. *Let P be a prime ideal of a ring R, and let I be a P-primary ideal of R such that $P^n \subseteq I$ for some positive integer n (for example, if R is a Noetherian ring). Then I is an n-absorbing ideal of R. Moreover, $\omega(I) \leq n$. In particular, if P^n is a P-primary ideal of R, then P^n is an n-absorbing ideal of R with $\omega(P^n) \leq n$, and $\omega(P^n) = n$ if $P^{n+1} \subset P^n$.*

Proof. Let $x_1 \cdots x_{n+1} \in I$ for $x_1, \ldots, x_{n+1} \in R$. If one of the x_i's is not in P, then the product of the other x_i's is in I since I is P-primary. Thus we may assume that every x_i is in P. Since $P^n \subseteq I$, we have $x_1 \cdots x_n \in I$. Hence I is an n-absorbing ideal of R.

The "moreover" and first part of the "in particular" statements are clear. The fact that $\omega(P^n) = n$ if $P^{n+1} \subset P^n$ follows from the proof of the "moreover" statement in Lemma 2.8. □

The hypothesis that $P^n \subseteq I$ for some positive integer n is needed in the above theorem since a primary ideal need not be an n-absorbing ideal for any positive integer n, see Example 5.6(a). Conversely, an n-absorbing ideal I with $Rad(I) = P$ a prime ideal need not be a P-primary ideal since every ideal in a Noetherian ring is an n-absorbing ideal for some positive integer n (Theorem 5.3), but an ideal with prime radical in a Noetherian ring need not be primary [10, Exercises 11 and 12, pp. 56–57]. In [3, Example 3.11], an example is given of a prime ideal P of a ring R such that P^2 is a 2-absorbing ideal of R, but P^2 is not P-primary (also, see Example 4.11(d)). We next give a sufficient condition for an n-absorbing ideal to be primary.

Theorem 3.2. *Let P be a divided prime ideal of a ring R, and let I be an n-absorbing ideal of R with $Rad(I) = P$. Then I is a P-primary ideal of R.*

Proof. Let $xy \in I$ for $x, y \in R$ and $y \notin P$. Then $x \in P$, and thus $x = y^{n-1}z$ for some $z \in R$ since $P \subset y^{n-1}R$ because P is a divided prime ideal of R and $y^{n-1} \notin P$. As $y^n z = yx \in I$, $y^n \notin I$, and I is an n-absorbing ideal of R, we have $x = y^{n-1}z \in I$. Hence I is a P-primary ideal of R. □

A special case of the next result is when P is a nonzero divided prime ideal in an integral domain R.

Theorem 3.3. *Let $Nil(R) \subset P$ be divided prime ideals of a ring R. Then P^n is a P-primary ideal of R, and thus P^n is an n-absorbing ideal of R with $\omega(P^n) \leq n$, for every positive integer n. Moreover, $\omega(P^n) = n$ if $P^{n+1} \subset P^n$.*

Proof. We show that P^n is a P-primary ideal of R. Then P^n is also an n-absorbing ideal of R by Theorem 3.1. Note that $Nil(R) \subset P^n$ since $Nil(R)$ is a divided prime ideal of R and $Nil(R) \subset P$. Let $xy \in P^n$ for $x, y \in R$ and $y \notin Rad(P^n) = P$. Then $xy = \sum z_{i1} \cdots z_{in}$ with each $z_{ij} \in P$. Since $P \subset yR$ because P is a divided prime ideal of R, each $z_{ij} = z'_{ij}y$ with $z'_{ij} \in P$. Thus $xy = zy$ with $z \in P^n$. Then $y(x - z) = 0 \in Nil(R)$ implies $x - z \in Nil(R) \subset P^n$. Hence $x \in P^n$ as desired.

The "moreover" statement follows from Theorem 3.1. □

Let I be a proper ideal of a ring R. For $x \in R$, let $I_x = \{y \in R \mid yx \in I\} = (I :_R x)$. We next investigate when I_x is an n-absorbing ideal of R. In particular, $\omega(I_x) \leq \omega(I)$ by Theorem 3.4. Example 4.11(b) and (c) show that this inequality may be strict for $x \in Rad(I)\backslash I$. These results generalize corresponding results for 2-absorbing ideals in [3].

Theorem 3.4. *Let I be an n-absorbing ideal of a ring R. Then $I_x = (I :_R x)$ is an n-absorbing ideal of R containing I for all $x \in R\backslash I$. Moreover, $\omega(I_x) \leq \omega(I)$ for all $x \in R$.*

Proof. Let $a_1 \cdots a_{n+1} \in I_x$ for $a_1, \ldots, a_{n+1} \in R$. Then $(xa_1)a_2 \cdots a_{n+1} \in I$, and thus either $a_2 \cdots a_{n+1} \in I$ or the product of xa_1 with $n-1$ of the a_i's for $2 \leq i \leq n+1$ is in I. In either case, there is a product of n of the a_i's that is in I_x. Thus I_x is an n-absorbing ideal of R. Clearly, $I \subseteq I_x$.

The "moreover" statement is clear if $x \in R\backslash I$ by above. If $x \in I$, then $I_x = R$, and hence $\omega(I_x) = 0 \leq \omega(I)$. □

Theorem 3.5. *Let $n \geq 2$ and $I \subset Rad(I)$ be an n-absorbing ideal of a ring R. Suppose that $x \in Rad(I)\backslash I$, and let $m(\geq 2)$ be the least positive integer such that $x^m \in I$. Then $I_{x^{m-1}} = (I :_R x^{m-1})$ is an $(n - m + 1)$-absorbing ideal of R containing I.*

Proof. First note that $2 \leq m \leq n$ since I is an n-absorbing ideal of R; so $n - m + 1 \geq 1$. Clearly $I \subseteq I_{x^{m-1}}$. Let $a_1 \cdots a_{n-m+2} \in I_{x^{m-1}}$ for $a_1, \ldots, a_{n-m+2} \in R$. Since $x^{m-1}a_1 \cdots a_{n-m+2} \in I$ and I is an n-absorbing ideal of R, either the product of x^{m-1} with some $n - m + 1$ of the a_i's is in I or $x^{m-2}a_1 \cdots a_{n-m+2} \in I$. If the product of x^{m-1} with some $n - m + 1$ of the a_i's is in I, then we are done. Hence assume that the product of x^{m-1} with any $n - m + 1$ of the a_i's is not in I, and thus $x^{m-2}a_1 \cdots a_{n-m+2} \in I$. Since $xx^{m-2}a_1 \cdots a_{n-m+1}(a_{n-m+2} + x) \in I$ and the product of

x^{m-1} with any $n-m+1$ of the a_i's is not in I, we must have $x^{m-2}a_1\cdots a_{n-m+2}+x^{m-1}a_1\cdots a_{n-m+1}=x^{m-2}a_1\cdots a_{n-m+1}(a_{n-m+2}+x)\in I$. As $x^{m-2}a_1\cdots a_{n-m+2}\in I$, we have $x^{m-1}a_1\cdots a_{n-m+1}\in I$, a contradiction since we assumed that the product of x^{m-1} with any $n-m+1$ of the a_i's is not in I. Thus the product of x^{m-1} with some $n-m+1$ of the a_i's is in I, and hence $I_{x^{m-1}}$ is an $(n-m+1)$-absorbing ideal of R containing I. □

Corollary 3.6. *Let $n\geq 2$ and $I\subset Rad(I)$ be an n-absorbing ideal of a ring R. Suppose that $x\in Rad(I)\backslash I$ and $x^n\in I$, but $x^{n-1}\notin I$. Then $I_{x^{n-1}}=(I:_R x^{n-1})$ is a prime ideal of R containing $Rad(I)$.*

Proof. Note that $I_{x^{n-1}}$ is an $(n-n+1)$-absorbing ideal of R containing I by Theorem 3.5, and thus $I_{x^{n-1}}$ is a prime ideal of R containing $Rad(I)$. □

Corollary 3.7. *Let $n\geq 2$ and I be an n-absorbing P-primary ideal of a ring R for some prime ideal P of R. If $x\in Rad(I)\backslash I$ and n is the least positive integer such that $x^n\in I$, then $I_{x^{n-1}}=(I:_R x^{n-1})=P$.*

Proof. By Corollary 3.6, we have $P=Rad(I)\subseteq I_{x^{n-1}}$. Let $y\in I_{x^{n-1}}$; so $x^{n-1}y\in I$. Since I is a P-primary ideal and $x^{n-1}\notin I$, we have $y\in P$. Thus $I_{x^{n-1}}=P$. □

The next two theorems concern when $(I:_R x)$ contains a subproduct of the minimal prime ideals of I.

Theorem 3.8. *Let $n\geq 2$ and $I\subset Rad(I)$ be an n-absorbing ideal of a ring R such that I has exactly n minimal prime ideals, say $P_1,\ldots,P_n$. Suppose that $x\in Rad(I)\backslash I$, and let $m(\geq 2)$ be the least positive integer such that $x^m\in I$. Then every product of $n-m+1$ of the P_i's is contained in $I_{x^{m-1}}=(I:_R x^{m-1})$.*

Proof. Note that $m\leq n$; so $n-m+1\geq 1$. Let $F=\{Q_1,\ldots,Q_{m-1}\}\subset G=\{P_1,\ldots,P_n\}$ and $D=G\backslash F$. Then D contains exactly $n-m+1$ of the P_i's. Since $x\in Rad(I)\backslash I$, we have $x\in Q_i$ for every $1\leq i\leq m-1$. Since $x^{m-1}\in Q_1\cdots Q_{m-1}$ and $(\prod_{Q\in F}Q)(\prod_{P\in D}P)=P_1\cdots P_n\subseteq I$ by Theorem 2.14, we have $x^{m-1}\prod_{P\in D}P\subseteq I$, and thus $\prod_{P\in D}P\subseteq I_{x^{m-1}}$. □

The proof of the following result is similar to that of Theorem 3.8.

Theorem 3.9. *Let $n\geq 2$ and $I\subset Rad(I)$ be an n-absorbing ideal of a ring R such that I has exactly n minimal prime ideals, say $P_1,\ldots,P_n$. If $x\in Rad(I)\backslash I$, then every product of $n-1$ of the P_i's is contained in $I_x=(I:_R x)$.*

Note that the ideal I in the next result is an n-absorbing ideal of R by Theorem 3.1.

Theorem 3.10. *Let I be a P-primary ideal of a ring R such that $P^n\subseteq I$ for some positive integer n (for example, if R is a Noetherian ring), and let $x\in P\backslash I$. If $x^m\notin I$ for some positive integer m, then $(I:_R x^m)=I_{x^m}$ is an $(n-m)$-absorbing ideal of R.*

Proof. First note that $m < n$ since $P^n \subseteq I$; so $n - m \geq 1$. Clearly, I_{x^m} is a P-primary ideal of R. We have $x^m P^{n-m} \subseteq I$ since $P^n \subseteq I$, and thus $P^{n-m} \subseteq I_{x^m}$. Hence I_{x^m} is an $(n-m)$-absorbing ideal of R by Theorem 3.1. □

4. EXTENSIONS OF *n*-ABSORBING IDEALS

In this section, we investigate the stability of n-absorbing ideals in various ring-theoretic constructions. The first two theorems and corollary generalize well-known results about prime ideals and follow directly from the definitions; so their proofs are omitted.

Theorem 4.1. *Let I be an n-absorbing ideal of a ring R, and let S be a multiplicatively closed subset of R with $I \cap S = \emptyset$. Then I_S is an n-absorbing ideal of R_S. Moreover, $\omega_{R_S}(I_S) \leq \omega_R(I)$.*

Theorem 4.2. *Let $f : R \longrightarrow T$ be a homomorphism of rings.*

(a) *Let J be an n-absorbing ideal of T. Then $f^{-1}(J)$ is an n-absorbing ideal of R. Moreover, $\omega_R(f^{-1}(J)) \leq \omega_T(J)$.*
(b) *Let f be surjective and I be an n-absorbing ideal of R containing* $\ker(f)$. *Then $f(I)$ is an n-absorbing ideal of T if and only if I is an n-absorbing ideal of R. Moreover, $\omega_T(f(I)) = \omega_R(I)$. In particular, this holds if f is an isomorphism.*

Corollary 4.3.

(a) *Let $R \subseteq T$ be an extension of rings and J an n-absorbing ideal of T. Then $J \cap R$ is an n-absorbing ideal of R. Moreover, $\omega_R(J \cap R) \leq \omega_T(J)$.*
(b) *Let $I \subseteq J$ be ideals of a ring R. Then J is an n-absorbing ideal of R if and only if J/I is an n-absorbing ideal of R/I. Moreover, $\omega_{R/I}(J/I) = \omega_R(J)$.*

We have seen in Example 2.7 that the product of n prime ideals of a ring R need not be an n-absorbing ideal of R. However, we do having the following result.

Corollary 4.4. *Let $P_1, \ldots, P_m$ be incomparable prime ideals of a ring R, $I = P_1^{n_1} \cdots P_m^{n_m}$ for positive integers $n_1, \ldots, n_m$ with $n = n_1 + \cdots + n_m$, and $S = R \backslash (P_1 \cup \cdots \cup P_m)$. Then $S(I) = \{x \in R \mid x/1 \in I_S\}$ is an n-absorbing ideal of R. In particular, $P^{(n)}$ is an n-absorbing ideal of R for P a prime ideal of R. Moreover, $\omega(S(I)) \leq \omega(I)$ and $\omega(P^{(n)}) \leq \omega(P^n)$.*

Proof. Let $f : R \longrightarrow R_S$ be the natural homomorphism $f(x) = x/1$. Then $(P_1)_S, \ldots, (P_m)_S$ are maximal ideals of R_S, and thus $I_S = (P_1^{n_1} \cdots P_m^{n_m})_S$ is an n-absorbing ideal of R_S by Theorem 2.9. Hence $S(I) = f^{-1}((P_1^{n_1} \cdots P_m^{n_m})_S)$ is an n-absorbing ideal of R by Theorem 4.2(a).

The "in particular" statement is clear since $P^{(n)} = S(P^n)$. For the "moreover" statement, note that $\omega_R(S(I)) \leq \omega_{R_S}(I_S) \leq \omega_R(I)$ by Theorem 4.2(a) and Theorem 4.1, respectively. Thus we also have $\omega(P^{(n)}) \leq \omega(P^n)$. □

The next corollary generalizes [2, Corollary 3.2], which gave the special case when $I = (P_1 \cdots P_n)_v$ is a principal ideal of a Krull domain R. A consequence of the following corollary is that every proper divisorial ideal of a Krull domain is an n-absorbing ideal for some positive integer n.

Corollary 4.5. *Let R be a Krull domain and $P_1, \ldots, P_n$ be ht-one prime ideals of R. Then $I = (P_1 \cdots P_n)_v$ is an n-absorbing ideal of R. Moreover, $\omega(I) = n$.*

Proof. Let $P_1 \cdots P_n = Q_1^{n_1} \cdots Q_k^{n_k}$ for distinct height-one prime ideals $Q_1, \ldots, Q_k$ of R and positive integers $n_1, \ldots, n_k$ with $n = n_1 + \cdots + n_k$. Then $I = (Q_1^{n_1} \cdots Q_k^{n_k})_v = Q_1^{(n_1)} \cap \cdots \cap Q_k^{(n_k)}$ by [6, Corollary 5.7, p. 26], and thus I is an n-absorbing ideal of R by Corollary 4.4 and Theorem 2.1(c).

For the "moreover" statement, we have $\omega(I) \leq n$ by above. Let $S = R \backslash (Q_1 \cup \cdots \cup Q_k)$. Then R_S is a principal ideal domain (PID) [6, Corollary 13.4, p. 58] and I_S is the product of n principal prime ideals of R_S; so $\omega_{R_S}(I_S) = n$. Hence $n \leq \omega(I)$ by Theorem 4.1, and thus $\omega(I) = n$. ☐

The next example shows that the inequalities in the above results may be strict.

Example 4.6. (a) The inequality in Theorem 4.1 may be strict. Let the ring R and the ideals I and I_n be as in Example 2.3. Note that R_M is a field for every maximal ideal M of R since R is a von Neumann regular ring. Since I and each I_n are proper ideals of R, we have $I \subseteq M$ and each $I_n \subseteq M_n$ for maximal ideals M and M_n of R. Thus $\omega_{R_M}(I_{nM_n}) = 1 < n = \omega_R(I_n)$ for each integer $n \geq 2$ and $\omega_{R_M}(I_M) = 1 < \infty = \omega_R(I)$. Also, let $J = 0$. Then $\omega(J) = \infty > 1 = \sup\{\omega(J_M) \mid M \in \mathrm{Spec}(R)\}$, i.e., a locally prime ideal need not be an n-absorbing ideal for any positive integer n.

(b) The inequality in Theorem 4.2(a) (and Corollary 4.3(a)) may be strict. Let $R = \mathbb{Q}[X] \subset T = \mathbb{Q}[X, Y]$ and $J = (X, Y^2)$ be an ideal of T. Then $\omega_T(J) = 2$ by Theorem 3.1. However, $J \cap R = XR$; so $\omega_R(J \cap R) = 1 < 2 = \omega_T(J)$.

(c) In Theorem 4.2(b), it is necessary to assume that $\ker(f) \subseteq I$. Let $R = \mathbb{Q}[X, Y]$, $T = \mathbb{Q}[X]$, and $f : R \longrightarrow T$ be the surjective ring homomorphism given by $f(g(X, Y)) = g(X, 0)$ for all $g(X, Y) \in R$. For $I_1 = (X^2 + Y)R$, we have $f(I_1) = X^2T$, and thus $\omega_R(I_1) = 1 < 2 = \omega_T(f(I_1))$. For $I_2 = (X, Y^2)$, we have $f(I_2) = XT$, and hence $\omega_R(I_2) = 2 > \omega_T(f(I_2)) = 1$. Note that $\ker(f) = YR$ is not contained in either I_1 or I_2.

(d) The inequalities in Corollary 4.4 may also be strict. Let the ring R, the prime ideals P_1 and P_2, and $I = P_1P_2$ be as in Example 2.7(a). Then $\omega(S(I)) = 2 < \omega(I)$ and $\omega(P_1^{(2)}) = 2 < \omega(P_1^2)$.

We next determine the n-absorbing ideals in the product of two, and hence any finite number of, rings. This generalizes the well-known result that the prime ideals of $R_1 \times R_2$ have the form $R_1 \times P_2$ or $P_1 \times R_2$ for P_i a prime ideal of R_i. Recall that an ideal of $R_1 \times R_2$ has the form $I_1 \times I_2$ for ideals I_i of R_i.

Theorem 4.7. *Let I_1 be an m-absorbing ideal of a ring R_1 and I_2 an n-absorbing ideal of a ring R_2. Then $I_1 \times I_2$ is an $(m + n)$-absorbing ideal of the ring $R_1 \times R_2$. Moreover, $\omega_{R_1 \times R_2}(I_1 \times I_2) = \omega_{R_1}(I_1) + \omega_{R_2}(I_2)$.*

Proof. Let $T = R_1 \times R_2$; we show that $\omega_T(I_1 \times I_2) = \omega_{R_1}(I_1) + \omega_{R_2}(I_2)$. First suppose that $\omega_{R_1}(I_1) = m < \infty$ and $\omega_{R_2}(I_2) = n < \infty$ (we may assume that $m, n \geq 1$). Then there are $x_1, \ldots, x_m \in R_1$ and $y_1, \ldots, y_n \in R_2$ such that $x_1 \cdots x_m \in I_1$ and $y_1 \cdots y_n \in I_2$, but no proper subproduct of the x_i's is in I_1 and no proper subproduct of the y_j's is in I_2. Thus $(x_1, 1) \cdots (x_m, 1)(1, y_1) \cdots (1, y_n) = (x_1 \cdots x_m, y_1 \cdots y_n) \in I_1 \times I_2$, but no proper subproduct is in $I_1 \times I_2$. Hence $\omega_T(I_1 \times I_2) \geq m + n = \omega_{R_1}(I_1) + \omega_{R_2}(I_2)$. Next, let $N = m + n + 1$ and suppose that $(x_1, y_1) \cdots (x_N, y_N) \in I_1 \times I_2$ for $(x_i, y_j) \in T$. Then $x_1 \cdots x_N \in I_1$ and $y_1 \cdots y_N \in I_2$; so there are $\{i_1, \ldots, i_m\}, \{j_1, \ldots, j_n\} \subset \{1, \ldots, N\}$ such that $x_{i_1} \cdots x_{i_m} \in I_1$ and $y_{j_1} \cdots y_{j_n} \in I_2$. Let $K = \{i_1, \ldots, i_m\} \cup \{j_1, \ldots, j_n\}$; so $|K| \leq m + n$. Thus $\prod_{k \in K}(x_k, y_k) \in I_1 \times I_2$; so $\omega_T(I_1 \times I_2) \leq m + n = \omega_{R_1}(I_1) + \omega_{R_2}(I_2)$. The above proof also shows that $\omega_T(I_1 \times I_2)$ is infinite if and only if either $\omega_{R_1}(I_1)$ or $\omega_{R_2}(I_2)$ is infinite. Hence $\omega_T(I_1 \times I_2) = \omega_{R_1}(I_1) + \omega_{R_2}(I_2)$. □

Corollary 4.8. *Let I_k be an ideal of a ring R_k for each integer $1 \leq k \leq n$, and let $R = R_1 \times \cdots \times R_n$. Then $\omega_R(I_1 \times \cdots \times I_n) = \omega_{R_1}(I_1) + \cdots + \omega_{R_n}(I_n)$.*

Corollary 4.9. *Let $I_1, \ldots, I_n$ be pairwise comaximal ideals of a ring R. Then $\omega(I_1 \cap \cdots \cap I_n) = \omega(I_1 \cdots I_n) = \omega(I_1) + \cdots + \omega(I_n)$. In particular, $\omega(M_1^{n_1} \cdots M_k^{n_k}) = \omega(M_1^{n_1}) + \cdots + \omega(M_k^{n_k})$ for distinct maximal ideals $M_1, \ldots, M_k$ of R and positive integers $n_1, \ldots, n_k$.*

Proof. It is sufficient to do the $n = 2$ case; so let I and J be comaximal ideals of R. Since I and J are comaximal, we have $R/IJ \cong R/I \times R/J$ by the Chinese Remainder Theorem and $IJ = I \cap J$. Thus $\omega_R(I \cap J) = \omega_R(IJ) = \omega_{R/IJ}(0) = \omega_{R/I \times R/J}(0 \times 0) = \omega_{R/I}(0) + \omega_{R/J}(0) = \omega_R(I) + \omega_R(J)$ by Corollary 4.3 and Theorem 4.7.

The "in particular" statement is clear. □

Let R be a ring, M be an R-module, and $T = R(+)M$. If I is an n-absorbing ideal of R, then it is easy to show that $I(+)M$ is an n-absorbing ideal of T. In fact, $\omega_T(I(+)M) = \omega_R(I)$. We have the following result for the special case $T = R(+)R$, where R is an integral domain.

Theorem 4.10. *Let D be an integral domain, $R = D(+)D$, and I be an n-absorbing ideal of D that is not an $(n-1)$-absorbing ideal of D. Then $0(+)I$ is an $(n+1)$-absorbing ideal of R that is not an n-absorbing ideal of R; so $\omega_R(0(+)I) = \omega_D(I) + 1$. In particular, if P is a prime ideal of D, then $0(+)P$ is a 2-absorbing ideal of R.*

Proof. Since I is an n-absorbing ideal of D that is not an $(n-1)$-absorbing ideal of D, there are $d_1, \ldots, d_n \in D$ such that $d_1 \cdots d_n \in I$ and no product of $n-1$ of the d_i's is in I. Let $b_1 = (d_1, 0), \ldots, b_n = (d_n, 0)$, and $b_{n+1} = (0, 1)$. Then $b_1 \cdots b_{n+1} = (0, d_1 \cdots d_n) \in 0(+)I$, and it is clear the no product of n of the b_i's is in $0(+)I$. Thus $0(+)I$ is not an n-absorbing ideal of R. Next we show that $0(+)I$ is an $(n+1)$-absorbing ideal of R. Let $c_1 = (a_1, m_1), \ldots, c_{n+2} = (a_{n+2}, m_{n+2}) \in R$ such that $c_1 \cdots c_{n+2} \in 0(+)I$. Since $Rad(0(+)I) = 0(+)D$ is a prime ideal of R, at least one of the c_i's is in $0(+)D$, say $c_1 = (0, m_1) \in 0(+)D$. Hence $c_1 \cdots c_{n+2} = (0, m_1 a_2 \cdots a_{n+2}) \in 0(+)I$, and thus $m_1 a_2 \cdots a_{n+2} \in I$. Hence either the product of m_1 with $n-1$ of the a_i's is in I or the product of n of the a_i's is in I; so either the product of c_1 with $n-1$ of the c_i's ($i \neq 1$) is in $0(+)I$ or the product of n of the c_i's ($i \neq 1$) is in $0(+)I$. Thus $0(+)I$ is an $(n+1)$-absorbing ideal of R, and hence $\omega_R(0(+)I) = n+1$.

The "in particular" statement is clear. □

The following example illustrates the previous theorem.

Example 4.11. Let $R = \mathbb{Z}(+)\mathbb{Z}$.

(a) Let $I = p_1 \cdots p_n\mathbb{Z}$, where $p_1, \ldots, p_n \in \mathbb{Z}$ are (not necessarily distinct) positive primes. Then I is an n-absorbing ideal of $\mathbb{Z}$ that is not an $(n-1)$-absorbing ideal of $\mathbb{Z}$. Thus $0(+)I$ is an $(n+1)$-absorbing ideal of R that is not an n-absorbing ideal of D by Theorem 4.10; so $\omega_R(0(+)I) = \omega_{\mathbb{Z}}(I) + 1 = n+1$.

(b) Let $p \in \mathbb{Z}$ be a positive prime. Then $J = 0(+)p\mathbb{Z}$ is a 2-absorbing ideal of R by Theorem 4.10 with $Rad(J) = 0(+)\mathbb{Z}$. For every $x \in Rad(J)\backslash J$, we have $J_x = (J :_R x) = (p\mathbb{Z})(+)\mathbb{Z}$, a prime ideal of R. Thus $\omega(J_x) = 1 < 2 = \omega(J)$ for every $x \in Rad(J)\backslash J$.

(c) If I is a 2-absorbing ideal of a ring T and $x, y \in Rad(I)\backslash I$, then I_x and I_y are linearly ordered prime ideals of T by [3, Theorems 2.5 and 2.6]. However, this need not be true if I is an n-absorbing ideal of a ring T and $n \geq 3$. Let $p_1, p_2 \in \mathbb{Z}$ be distinct positive primes. Then $J = 0(+)p_1p_2\mathbb{Z}$ is a 3-absorbing ideal of R that is not a 2-absorbing ideal of $R = \mathbb{Z}(+)\mathbb{Z}$ by Theorem 4.10 with $Rad(J) = 0(+)\mathbb{Z}$. Let $x = (0, n) \in Rad(J)\backslash J$. Then $J_x = (p_1p_2\mathbb{Z})(+)\mathbb{Z}$ and $\omega(J_x) = 2$ if $n \in \mathbb{Z}\backslash(p_1\mathbb{Z} \cup p_2\mathbb{Z})$, $J_x = (p_1\mathbb{Z})(+)\mathbb{Z}$ and $\omega(J_x) = 1$ if $n \in p_2\mathbb{Z}\backslash p_1p_2\mathbb{Z}$, and $J_x = (p_2\mathbb{Z})(+)\mathbb{Z}$ and $\omega(J_x) = 1$ if $n \in p_1\mathbb{Z}\backslash p_1p_2\mathbb{Z}$, and these ideals are not linearly ordered. Thus, $\omega(J) = 3$, while $\omega(J_x)$ is either 1 or 2 for all $x \in Rad(J)\backslash J$.

(d) Let I be a P-primary ideal of a ring T such that $P^m \subseteq I$ for some positive integer m. If I is an n-absorbing ideal of T that is not an $(n-1)$-absorbing ideal of T, then $\omega_T(I) = n \leq m$ by Theorem 3.1. However, by (a) above, for every integer $n \geq 2$, the ring $R = \mathbb{Z}(+)\mathbb{Z}$ and ideal $I_n = 0(+)(p_1 \cdots p_{n-1}\mathbb{Z})$ of R have $P = Rad(I_n) = 0(+)\mathbb{Z}$ a prime ideal of R such that $P^2 \subseteq I_n$ and $\omega_R(I_n) = n$ (so I_n is not P-primary when $n \geq 3$).

Let T be a ring extension of an integral domain D and P a prime ideal of D. Then $0(+)P$ need not be a 2-absorbing ideal of the ring $R = D(+)T$; so Theorem 4.10 does not extend to general R. We have the following example.

Example 4.12. Let $R = \mathbb{Z}(+)\mathbb{Q}$. Then $I = 0(+)2\mathbb{Z}$ is an ideal of R with $Rad(I) = 0(+)\mathbb{Q}$. Let $x = (0, \frac{1}{2}) \in Rad(I)\backslash I$. Then $I_x = (I :_R x) = (4\mathbb{Z})(+)\mathbb{Q}$ is not a prime ideal of R ($\omega(I_x) = 2$), and hence I is not a 2-absorbing ideal of R by [3, Theorem 2.8]. In fact, one can easily show that I is not an n-absorbing ideal of R for any positive integer n. For each positive integer n, let $x_i = (2, 0)$ for $1 \leq i \leq n$ and $x_{n+1} = (0, \frac{1}{2^{n-1}})$. Then $x_1 \cdots x_{n+1} = (0, 2) \in I$, but no proper subproduct of the x_i's is in I. Thus $\omega_R(I) = \infty$.

We next briefly consider extensions of n-absorbing ideals of R in the polynomial ring $R[X]$.

Theorem 4.13. *Let I be an ideal of a ring R. Then (I, X) is an n-absorbing ideal of $R[X]$ if and only if I is an n-absorbing ideal of R. Moreover, $\omega_{R[X]}((I, X)) = \omega_R(I)$.*

Proof. This follows directly from Corollary 4.3(b) since $(I, X)/(X) \cong I$ in $R[X]/(X) \cong R$.

The "moreover" statement is clear. $\square$

It is also natural to ask if $\omega_{R[X]}(I[X]) = \omega_R(I)$. This is well known if I is a prime ideal of R, and we conjecture that the equality holds for all ideals I of R. While we have been unable to prove the general result, we next show that I is a 2-absorbing ideal of R if and only if $I[X]$ is a 2-absorbing ideal of $R[X]$. But first, we need the following trivial lemma.

Lemma 4.14. *Let I be an ideal of a ring R and suppose that $x + y \in I$ for some $x, y \in R$. Then $I_x = I_y$ (recall that $I_x = (I :_R x)$).*

Theorem 4.15. *Let I be an ideal of a ring R. Then $I[X]$ is a 2-absorbing ideal of $R[X]$ if and only if I is a 2-absorbing ideal of R.*

Proof. If $I[X]$ is a 2-absorbing ideal of $R[X]$, then I is a 2-absorbing ideal of R by Corollary 4.3(a).

Conversely, suppose that I is a 2-absorbing ideal of R. Recall that $I[X]$ is a prime ideal of $R[X]$ if and only if I is a prime ideal of R. Thus we may assume that I is not a prime ideal of R. Since either $Rad(I) = P$ is a prime ideal of R or $Rad(I) = P_1 \cap P_2$ for prime ideals P_1, P_2 of R with $P_1P_2 \subseteq I$ by [3, Theorem 2.4], we conclude that either $Rad(I[X]) = P[X]$ or $Rad(I[X]) = P_1[X] \cap P_2[X]$ with $P_1[X]P_2[X] \subseteq I[X]$. Now let $f(X) = a_nX^n + \cdots + a_0 \in Rad(I[X]) \backslash I[X]$. By [3, Theorems 2.8 and 2.9], it suffices to show that $I[X]_{f(X)}$ is a prime ideal of $R[X]$. Without loss of generality, we may assume that $a_i \notin I$ for all $0 \le i \le n$. Since $I_{a_0}, I_{a_1}, \ldots, I_{a_n}$ are linearly ordered prime ideals of R by [3, Theorems 2.5 and 2.6], there is a k with $0 \le k \le n$ such that $I_{a_k} \subseteq I_{a_i}$ for every i, $0 \le i \le n$. We show that $I[X]_{f(X)} = I_{a_k}[X]$ is a prime ideal of $R[X]$. It is clear that $I_{a_k}[X] \subseteq I[X]_{f(X)}$. Let $g(X) = b_mX^m + \cdots + b_0 \in I[X]_{f(X)}$; so $f(X)g(X) = a_0b_0 + \cdots + a_nb_mX^{n+m} \in I[X]$. Since $b_0a_0 \in I$, we conclude that $b_0 \in I_{a_0}$. Now let $i < k$ such that $b_0 \in I_{a_c}$ for every c with $0 \le c \le i < k$; we will show that $b_0 \in I_{a_{i+1}}$. Consider the term $(b_{i+1}a_0 + b_ia_1 + \cdots + b_1a_i + b_0a_{i+1})X^{i+1}$ of $f(X)g(X)$ (observe that some of the b_j's might be zero). Let $t = b_{i+1}a_0 + \cdots + b_1a_i$. Since $t + b_0a_{i+1} \in I$, we have $I_t = I_{b_0a_{i+1}}$ by Lemma 4.14. Since $b_0 \in I_{a_c}$ for every c with $0 \le c \le i < k$, we have $b_0 \in I_t$, and thus $b_0 \in I_{b_0a_{i+1}}$. Hence $b_0 \in I_{a_{i+1}}$ because $I_{a_{i+1}}$ is a prime ideal of R, and thus $b_0 \in I_{a_k}$. Now suppose that $b_0, \ldots, b_h \in I_{a_k}$ for some h with $0 \le h < m$. We show that $b_{h+1} \in I_{a_k}$. Consider the term $(b_0a_{h+1} + \cdots + b_{h+1}a_0)X^{h+1}$ of $f(X)g(X)$; at once we conclude that $b_{h+1} \in I_{a_0}$. By repeating a similar argument to the one used earlier to show that $b_0 \in I_{a_k}$, we have $b_{h+1} \in I_{a_k}$. Thus $g(X) \in I_{a_k}[X]$. Hence $I[X]_{f(X)} = I_{a_k}[X]$ is a prime ideal of $R[X]$, and thus $I[X]$ is a 2-absorbing ideal of $R[X]$. $\square$

We conclude this section by investigating n-absorbing ideals for the "$D + M$" construction. Let $T = K + M$ be an integral domain, where K is a field which is a subring of T and M is a nonzero maximal ideal of T, and let D be a subring of K. Then $R = D + M$ is a subring of T with $qf(R) = qf(T)$. This construction has proved very useful for constructing examples (cf. [4, 5, 8, 11]). The first lemma describes the n-absorbing ideals of R which contain M.

Lemma 4.16. *Let $T = K + M$ be an integral domain, where K is a field which is a subring of T and M is a nonzero maximal ideal of T. Let D be a subring of K and $R = D + M$. Let I be an ideal of D. Then $I + M$ is an n-absorbing ideal of R if and only if I is an n-absorbing ideal of D. Moreover, $\omega_R(I + M) = \omega_D(I)$.*

Proof. This follows directly from Corollary 4.3(b) since $(I + M)/M \cong I$ in $R/M \cong D$.

The "moreover" statement is clear. □

To get more complete results, we restrict to the case where $T = K[[X]] = K + XK[[X]]$ for K a field and $M = XK[[X]]$. In this case, every ideal of $R = D + M = D + XK[[X]]$ is comparable to M. The ideals of R which contain M have the form $I + XK[[X]]$ for I an ideal of D, and the ideals contained in M have the form $WX^n + X^{n+1}K[[X]]$ for W a D-submodule of K and n a positive integer [4, Theorem 2.1]. Note that $M^n = X^nK[[X]]$ has $\omega_T(M^n) = \omega_R(M^n) = n$ for every positive integer n.

Theorem 4.17. *Let D be a subring of a field K and $R = D + XK[[X]]$.*

(a) *If D is a field, then every proper ideal of R is an n-absorbing ideal of R for some positive integer n.*
(b) *If D is a proper subring of K with $qf(D) = K$, then the nonzero n-absorbing ideals of R have the form $I + XK[[X]]$, where I is an n-absorbing ideal of D, or $X^mK[[X]]$ for m an integer with $1 \leq m \leq n$. Moreover, $\omega_R(I + XK[[X]]) = \omega_D(I)$ and $\omega_R(X^mK[[X]]) = m$.*

Proof. (a) If $D = K$, this is clear since then $R = K[[X]]$ is a DVR. So, let $D = F$ be a proper subfield of K. Then $R = F + XK[[X]]$ is a one-dimensional quasilocal integral domain with maximal ideal $M = XK[[X]]$. Each proper nonzero ideal of R is M-primary and has the form $I = WX^n + X^{n+1}K[[X]]$ for some nonzero F-subspace W of K and positive integer n. Then $M^{n+1} \subseteq I$ implies that $\omega_R(I) \leq n + 1$ by Theorem 3.1. Thus every proper ideal of R is an n-absorbing ideal of R for some positive integer n.

(b) Let $D \subset qf(D) = K$ and J be a nonzero n-absorbing ideal of R. Then J is comparable to M. If $M \subset J$, then $J = I + XK[[X]]$ for I an n-absorbing ideal of D by Lemma 4.16. So we may assume that $J \subseteq M$. Then $J = WX^m + X^{m+1}K[[X]]$ for W a nonzero D-submodule of K and positive integer m. Suppose that $W \subset K$. Then there are $0 \neq a, d \in D$ such that $a \in W$, but $\frac{a}{d^i} \notin W$ for all positive integers i. Then $d^i(\frac{a}{d^i}X^m) = aX^m \in J$, but no proper subproduct is in J; so J is not an i-absorbing ideal of R for any positive integer i, i.e., $\omega_R(J) = \infty$. Thus $W = K$; so $J = X^mK[[X]]$ is an m-absorbing ideal of R and $1 \leq m \leq n$.

The "moreover" statement follows from Lemma 4.16 and the comments before this theorem. □

The next example illustrates the two cases of the previous theorem.

Example 4.18. (a) Let $R = \mathbb{Q} + X\mathbb{R}[[X]] \subset \mathbb{R}[[X]]$. Then R is a one-dimensional quasilocal integral domain with non-finitely generated maximal ideal $M = X\mathbb{R}[[X]]$, and R is not a valuation domain [8, Exercises 12–13, pp. 202–203]. Each proper nonzero ideal of R has the form $I = WX^n + X^{n+1}\mathbb{R}[[X]]$ for some nonzero $\mathbb{Q}$-subspace W of $\mathbb{R}$ and positive integer n. Then $M^{n+1} \subseteq I$ implies that $\omega_R(I) \leq n+1$. If $W = \mathbb{R}$, then $I = X^n\mathbb{R}[[X]]$ and $\omega_R(I) = n$; otherwise $\omega_R(I) = n+1$. To see this, let $\alpha \in \mathbb{R}\backslash W$ with $\alpha > 0$, and let $\beta = \alpha^{\frac{1}{n}}$. Then $(\beta X)^{n+1} \in I$, but $(\beta X)^n = \alpha X^n \notin I$. Thus every proper ideal of R is an n-absorbing ideal of R for some positive integer n.

(b) Let $R = F + XK[[X]]$ for F a proper subfield of K. Then R is a one-dimensional quasilocal integral domain with maximal ideal $M = XK[[X]]$. Moreover, R is never a valuation domain and ring-theoretic properties of R depend on the field extension K/F. For example, R is Noetherian if and only if $[K : F] < \infty$, and R is integrally closed if and only if F is algebraically closed in K [4, Theorem 2.1]. Thus for various choices of fields $F \subset K$, we obtain integral domains R satisfying certain ring-theoretic properties, and all proper ideals of R are n-absorbing ideals of R for some positive integer n by Theorem 4.17(a).

(c) Let $R = \mathbb{Z} + X\mathbb{Q}[[X]] \subset \mathbb{Q}[[X]]$. Then R is a two-dimensional Bézout domain which is not a valuation domain with $Spec(R) = \{0, X\mathbb{Q}[[X]]\} \cup \{pR \mid p \in \mathbb{Z} \text{ prime}\}$ [5, Theorem 7 and Corollary 9]. The ideal $I = XR = \mathbb{Z}X + X^2\mathbb{Q}[[X]]$ is not an n-absorbing ideal of R for any positive integer n; so $\omega_R(I) = \infty$. This follows since $2^n(\frac{1}{2^n}X) = X \in I$, but no proper subproduct is in I. By Theorem 4.17(b) (or Theorem 5.7), a nonzero n-absorbing ideal of R has the form $I_1 = p_1^{n_1} \cdots p_k^{n_k} R = p_1^{n_1} \cdots p_k^{n_k}\mathbb{Z} + X\mathbb{Q}[[X]]$ for distinct positive primes $p_1, \ldots, p_k \in \mathbb{Z}$ and positive integers $n_1, \ldots, n_k$ with $n_1 + \cdots + n_k \leq n$ or $I_2 = X^m\mathbb{Q}[[X]]$ for m a positive integer with $m \leq n$. Moreover, $\omega_R(I_1) = n_1 + \cdots + n_k$ and $\omega_R(I_2) = m$.

(d) Let D be a subring of a field K with $D \subset qf(D) = F \subset K$. Then the nonzero ideals of $R = D + M$ contained in $M = XK[[X]]$ have the form $I = WX^m + X^{m+1}K[[X]]$ for some nonzero D-submodule W of K and positive integer m. If $W \subset K$, then I may or may not be an n-absorbing ideal of R for some positive integer n. For example, let $I_1 = XR = DX + X^2K[[X]]$ and $I_2 = FX + X^2K[[X]]$. Then one can easily verify that $\omega_R(I_1) = \infty$ and $\omega_R(I_2) = 2$.

5. *n*-ABSORBING IDEALS IN SPECIFIC RINGS

In this section, we study n-absorbing ideals in several special classes of commutative rings. If R is a Dedekind domain, then every proper nonzero ideal of R is a product of maximal ideals of R, and hence is an n-absorbing ideal for some positive integer n (see Corollary 4.5 for a Krull domain analog). Specifically, if $I = M_1 \cdots M_n$ with each M_i a maximal ideal of R, then I is an n-absorbing ideal of R by Theorem 2.9. In fact, the converse is true if R is a Noetherian integral domain.

Theorem 5.1. *Let R be a Noetherian integral domain. Then the following statements are equivalent*:

(1) *R is a Dedekind domain*;
(2) *If I is an n-absorbing ideal of R, then $I = M_1 \cdots M_m$ for maximal ideals $M_1, \ldots, M_m$ of R with $1 \leq m \leq n$.*

Moreover, if $I = M_1 \cdots M_n$ for maximal ideals $M_1, \ldots, M_n$ of a Dedekind domain R which is not a field, then $\omega(I) = n$.

Proof. (1) $\Rightarrow$ (2) This has already been observed above.

(2) $\Rightarrow$ (1) Let M be a maximal ideal of R. Since every ideal between M^2 and M is an M-primary ideal of R, and hence a 2-absorbing ideal of R by Theorem 3.1, the hypothesis in (2) implies that there are no ideals of R properly between M^2 and M. Thus R is a Dedekind domain by [8, Theorem 39.2].

The "moreover" statement follows from Lemma 2.8 and Corollary 4.9. □

We next give a similar result for almost Dedekind domains.

Theorem 5.2. *Let R be an almost Dedekind domain. Then a nonzero ideal I of R is an n-absorbing ideal of R if and only if $I = M_1 \cdots M_m$ for maximal ideals $M_1, \ldots, M_m$ of R with $1 \leq m \leq n$. Moreover, $\omega(M_1 \cdots M_m) = m$.*

Proof. Let I be a nonzero n-absorbing ideal of R. Then there are only a finite number of prime (maximal) ideals of R minimal over I, say $P_1, \ldots, P_k$ with $k \leq n$, by Theorem 2.5. For each $1 \leq i \leq k$, we have $I_{P_i} = (P_i^{n_i})_{P_i}$ for some positive integer n_i since R_{P_i} is a DVR (note that $n_i \leq n$ by Theorem 4.1). Let $J = P_1^{n_1} \cdots P_k^{n_k}$. Then $I_M = J_M$ for each maximal ideal M of R; so $I = J$ is a product of maximal ideals of R. The converse holds by Theorem 2.9.

The "moreover" statement follows as in Theorem 5.1. □

We have seen that a ring may have proper ideals that are not n-absorbing ideals for any positive integer n. However, we next show that in a Noetherian ring, every proper ideal is an n-absorbing ideal for some positive integer n.

Theorem 5.3. *Let R be a Noetherian ring. Then every proper ideal of R is an n-absorbing ideal of R for some positive integer n.*

Proof. Let J be a P-primary ideal of R for some prime ideal P of R. Then $P^m \subseteq J$ for some positive integer m since R is Noetherian. Thus J is an m-absorbing ideal of R by Theorem 3.1. Let I be a proper ideal of R. Then I is a finite intersection of primary ideals of R since R is Noetherian [10, Theorem 2.7], and hence I is an n-absorbing ideal of R for some positive integer n by Theorem 2.1(c). □

We next determine the n-absorbing ideals in a valuation domain. We will need the following lemma (cf. Theorem 6.2).

Lemma 5.4. *Let R be a Bézout ring, I an n-absorbing ideal of R, and P a prime ideal of R such that $Rad(I) = P$. Then $P^n \subseteq I$. In particular, this holds if R is a valuation domain.*

Proof. Let $x_1, \ldots, x_n \in P$. Since R is a Bézout ring, we have $(x_1, \ldots, x_n) = xR$ for some $x \in P$, and thus $x_1 \cdots x_n = x^n y$ for some $y \in R$. Since $x^n \in I$ by Theorem 2.1(e), we have $x_1 \cdots x_n = x^n y \in I$, and hence $P^n \subseteq I$.

The "in particular" statement is clear. □

The key fact for our characterization of n-absorbing ideals in a valuation domain R is that if the prime ideal P of R is not idempotent, then every P-primary ideal of R has the form P^m for some positive integer m [8, Theorem 17.3(b)]. Since $PQ = P$ for prime ideals $P \subset Q$ of R, the following theorem may be restated as follows: an ideal I of a valuation domain R is an n-absorbing ideal of R for some positive integer n if and only if I is a product of prime ideals of R.

Theorem 5.5. *Let R be a valuation domain and n a positive integer. Then the following statements are equivalent for an ideal I of R:*

(1) *I is an n-absorbing ideal of R;*
(2) *I is a P-primary ideal of R for some prime ideal P of R and $P^n \subseteq I$;*
(3) *$I = P^m$ for some prime ideal $P(= Rad(I))$ of R and integer m with $1 \leq m \leq n$.*

Moreover, $\omega(P^n) = n$ for P a nonidempotent prime ideal of R.

Proof. (1) $\Rightarrow$ (2) Let I be an n-absorbing ideal of R. Then $P = Rad(I)$ is a divided prime ideal of R by [8, Theorem 17.1(2)], and hence I is a P-primary ideal of R by Theorem 3.2. We have $P^n \subseteq I$ by Lemma 5.4.

(2) $\Rightarrow$ (3) This follows from [8, Theorem 17.3(b)].

(3) $\Rightarrow$ (1) We may assume that I is nonzero. By Theorem 3.3, $I = P^m$ is an m-absorbing ideal of R. Thus I is also an n-absorbing ideal of R by Theorem 2.1(b).

The "moreover" statement follows from Theorem 3.3 since $P^{n+1} \subset P^n$ for every positive integer n. □

Our next example shows that the "Noetherian" hypothesis is needed in Theorems 5.1 and 5.3. In each case, the specific details follow directly from Theorem 5.5 and well-known results about the value group of a valuation domain (cf. [8]).

Example 5.6. (a) Let R be a one-dimensional valuation domain with maximal ideal M. Thus all nonzero proper ideals of R are M-primary. If M is principal, then R is a DVR, and thus every proper ideal of R is an n-absorbing ideal for some positive integer n. In this case, $\omega(M^n) = n$ and $\omega(0) = 1$. If M is not principal, then $M = M^2$, and hence 0 and M are the only n-absorbing ideals of R for any positive integer n. In this case, $\omega(M) = \omega(0) = 1$ and $\omega(I) = \infty$ for any ideal I of R with $0 \subset I \subset M$. Note that I is M-primary (cf. Theorem 3.1).

(b) Let R be a two-dimensional valuation domain with prime ideals $0 \subset P \subset M$ and value group G. If $G = \mathbb{Z} \oplus \mathbb{Z}$ (all direct sums have the lexicographic order), then $P^2 \neq P$ and $M^2 \neq M$; so $0, P^k$, and M^k with $1 \leq k \leq n$ are the only n-absorbing ideals of R for any positive integer n (i.e., $\omega(M^n) = \omega(P^n) = n$ and $\omega(0) = 1$). If $G = \mathbb{Q} \oplus \mathbb{Q}$, then $P^2 = P$ and $M^2 = M$; so $0, P$ and M are the only n-absorbing ideals of R for any positive integer n (i.e., $\omega(M) = \omega(P) = \omega(0) = 1$). If $G = \mathbb{Z} \oplus \mathbb{Q}$, then $M^2 = M$ and $P^2 \neq P$; so $0, P^k$ with $1 \leq k \leq n$, and M are the only n-absorbing ideals of R for any positive integer n (i.e., $\omega(M) = \omega(0) = 1$ and $\omega(P^n) = n$). If $G = \mathbb{Q} \oplus \mathbb{Z}$, then $P^2 = P$ and $M^2 \neq M$; so $0, P$, and M^k with $1 \leq k \leq n$ are the only n-absorbing ideals of R for any positive integer n (i.e., $\omega(P) = \omega(0) = 1$ and $\omega(M^n) = n$).

(c) For each positive integer m or ∞, there is a valuation domain R with $\dim(R) = m$ such that the prime ideals of R are the only n-absorbing ideals of R for any positive integer n (let R be a valuation domain with value group $G = \bigoplus_{i=1}^{m} \mathbb{Q}$).

These results can also be extended to Prüfer domains. Recall that incomparable prime ideals of a Prüfer domain R are comaximal since R is locally a valuation domain. Also, a prime ideal P of a Prüfer domain R is idempotent if and only if P_P is idempotent in R_P.

Theorem 5.7. *Let R be a Prüfer domain. Then an ideal I of R is an n-absorbing ideal of R for some positive integer n if and only if I is a product of prime ideals of R. Moreover, if $P_1, \ldots, P_k$ are incomparable prime ideals of R and $n_1, \ldots, n_k$ are positive integers with $n_i = 1$ if P_i is idempotent, then $\omega(P_1^{n_1} \cdots P_k^{n_k}) = n_1 + \cdots + n_k$.*

Proof. Let I be a nonzero n-absorbing ideal of R, and let $P_1, \ldots, P_k$ with $k \leq n$ be the minimal prime ideals of I (Theorem 2.5). Then the P_i's are pairwise comaximal since R is a Prüfer domain. By Theorems 4.1 and 5.5, we have $I_{P_i} = (P_i^{n_i})_{P_i}$ for some positive integer n_i. Let $J = P_1^{n_1} \cdots P_k^{n_k}$. Let M be a maximal ideal of R; we may assume that P_i is the only minimal prime ideal of I contained in M. As above, $I_M = (P_i^{k_i})_M$ for some positive integer k_i; and we can assume that $k_i = n_i$ since $(I_M)_{P_i} = I_{P_i}$. Thus $I_M = J_M$ for every maximal ideal M of R, and hence $I = J$. So I is a product of prime ideals of R.

Conversely, suppose that I is a product of prime ideals of R. Note that if $P \subset Q$ are prime ideals of R, then $PQ = P$ (since this holds locally). Thus we may assume that $I = P_1^{n_1} \cdots P_k^{n_k}$, where $P_1, \ldots, P_k$ are comaximal prime ideals of R and the n_i's are positive integers with $n = n_1 + \cdots + n_k$. Each $P_i^{n_i}$ is a P_i-primary ideal of R by [8, Lemma 23.2(b)]. Thus each $P_i^{n_i}$ is an n_i-absorbing ideal of R by Theorem 3.1, and hence I is an n-absorbing ideal of R by Theorem 2.1(c) since $I = P_1^{n_1} \cdots P_k^{n_k} = P_1^{n_1} \cap \cdots \cap P_k^{n_k}$ (or use Corollary 4.9).

The "moreover" statement follows from Theorem 3.1 and Corollary 4.9. □

We have seen that every proper ideal of either a Noetherian ring or certain valuation domains is an n-absorbing ideal for some positive integer n. For any ring R, we define $\Omega(R) = \{\omega_R(I) \mid I \text{ is a proper ideal of } R\}$. Then $\{1\} \subseteq \Omega(R) \subseteq \mathbb{N} \cup \{\infty\}$. The following example and theorems give the possible values for $\Omega(R)$ in several classes of rings.

Example 5.8. (a) Let $n = p_1^{n_1} \cdots p_k^{n_k}$ for distinct positive primes $p_1, \ldots, p_k \in \mathbb{Z}$ and positive integers $n_1, \ldots, n_k$. Then $\Omega(\mathbb{Z}_n) = \{1, \ldots, m\}$, where $m = n_1 + \cdots + n_k$, by Theorem 4.7. In particular, $\Omega(\mathbb{Z}_{p^n}) = \{1, \ldots, n\}$ for any positive prime $p \in \mathbb{Z}$ and positive integer n.

(b) Let $R = \mathbb{Z}$ (or any PID, not a field); then $\Omega(R) = \mathbb{N}$ by Theorem 2.1(d).

(c) Let $R = \prod_{i=1}^{\infty} \mathbb{Z}_2$, I, and I_n be as in Example 2.3. Then $\omega(I_n) = n$ for each $n \in \mathbb{N}$ and $\omega(I) = \infty$; so $\Omega(R) = \mathbb{N} \cup \{\infty\}$.

(d) Let R be a zero-dimensional quasilocal ring with maximal ideal M such that $M^{n+1} \subset M^n$ for every positive integer n (for example, let K be a field and $R = K[\{X_n \mid n \in \mathbb{N}\}]/(\{X_n^{n+1} \mid n \in \mathbb{N}\}))$. Then $\mathbb{N} \subseteq \Omega(R)$ by Lemma 2.8 (cf. Remark 6.4(b)).

(e) Let $T = \mathbb{Q} + X\mathbb{R}[[X]]$, $M = X\mathbb{R}[[X]]$, and n be a positive integer. Then $\Omega(T) = \mathbb{N}$ and $R = T/M^n$ has $\Omega(R) = \{1, \ldots, n\}$ by Example 4.18(a) and Corollary 4.3(b). Note that T is not Noetherian and R is not Artinian for $n \geq 2$.

Let n be a fixed positive integer. Then it is easy to give necessary conditions for every proper ideal of a ring R to be an n-absorbing of R, i.e, $\Omega(R) \subseteq \{1, \ldots, n\}$. Example 5.8(d) shows that the converse of the following theorem is false: a quasilocal ring R with $\dim(R) = 0$ may have $\Omega(R)$ infinite. For a converse, see Theorem 6.5. Also, we may have $\Omega(R)$ finite and $\dim(R) > 0$ if $\infty \in \Omega(R)$ (Example 5.6(a)).

Theorem 5.9. *Let R be a ring and n a positive integer such that every proper ideal of R is an n-absorbing ideal of R. Then $\dim(R) = 0$ and R has at most n maximal ideals.*

Proof. Suppose that $\dim(R) \geq 1$; so R has prime ideals $P \subset Q$. Choose $x \in Q \backslash P$, and let $I = x^{n+1}R$. Then $x^n \in I$ since I is an n-absorbing ideal of R, and thus $x^n = x^{n+1}y$ for some $y \in R$. Hence $x^n(1 - xy) = 0 \in P$, and thus $1 - xy \in P \subset Q$. Then $x \in Q$ gives $1 \in Q$, a contradiction, and hence $\dim(R) = 0$. That R has at most n maximal ideals follows from Theorem 2.6. □

Lemma 5.10. *Let M be a finitely generated maximal ideal of a ring R. If $M^n = M^{n+1}$ for some positive integer n, then $ht(M) = 0$. In particular, if R is Noetherian with $\dim(R) \geq 1$, then there is a maximal ideal M of R with $M^{n+1} \subset M^n$ for every positive integer n.*

Proof. Suppose that $\mathrm{ht}(M) \geq 1$. Then $P \subset M$ for some prime ideal P of R. In R_M, we have $P_M \subset M_M$ and $M_M M_M^n = M_M^n$; so $M_M^n = 0$ by Nakayama's Lemma. But then $M_M^n \subseteq P_M$; so $P_M = M_M$, a contradiction. Thus $\mathrm{ht}(M) = 0$.

The "in particular" statement is clear. □

Theorem 5.11. *Let R, R_1, and R_2 be rings.*

(a) *If $|Max(R)| = n < \infty$, then $\{1, \ldots, n\} \subseteq \Omega(R)$. If $Max(R)$ is infinite, then $\mathbb{N} \subseteq \Omega(R)$.*
(b) *Let I be a proper ideal of R. Then $\Omega(R/I) \subseteq \Omega(R)$.*
(c) $\Omega(R) \subseteq \Omega(R[X])$.
(d) $\Omega(R_1 \times R_2) = \Omega(R_1) + \Omega(R_2)$.
(e) *Let M be an R-module. Then $\Omega(R) \subseteq \Omega(R(+)M)$.*
(f) *Let $T = K + M$ be an integral domain, where K is a field which is a subring of T and M is a nonzero maximal ideal of T, and let D be a subring of K. Then $\Omega(D) \subseteq \Omega(D + M)$.*
(g) *R is a field if and only if $\Omega(R) = \{1\}$.*
(h) *If R is an Artinian ring, then $\Omega(R) = \{1, \ldots, n\}$ for some $n \in \mathbb{N}$.*

(i) *If R is a Noetherian ring with* $\dim(R) \geq 1$, *then* $\Omega(R) = \mathbb{N}$.
(j) *Let R be a valuation domain (not a field). Then* $\Omega(R) = \mathbb{N}$ *if R is a DVR. If R is not a DVR, then* $\Omega(R) = \{1, \infty\}$ *if all nonzero prime ideals of R are idempotent, and* $\Omega(R) = \mathbb{N} \cup \{\infty\}$ *if R has a nonidempotent nonzero prime ideal.*

Proof. (a) This follows from Theorem 2.6.

(b) This follows from Corollary 4.3(b).

(c) This follows from Theorem 4.13.

(d) This follows from Theorem 4.7.

(e) This follows from our earlier observation just before Theorem 4.10 that $\omega_{R(+)M}(I(+)M) = \omega_R(I)$ for every ideal I of R.

(f) This follows from Lemma 4.16.

(g) This is clear since every proper ideal of a ring R is a prime ideal if and only if R is a field.

(h) If R is local, this follows from Theorem 3.1. The general case then follows from Corollary 4.8 since every Artinian ring is the direct product of finitely many local Artinian rings.

(i) We have $\Omega(R) \subseteq \mathbb{N}$ by Theorem 5.3. Lemmas 2.8 and 5.10 give $\mathbb{N} \subseteq \Omega(R)$. Thus $\Omega(R) = \mathbb{N}$.

(j) This follows from Theorem 5.5. □

The inclusions in the above theorem may be strict. This is clear for (a), (b), (e), and (f). For (c), let R be any Artinian ring. Then $\Omega(R)$ is finite by (h), but $\Omega(R[X]) = \mathbb{N}$ by (i) since $R[X]$ is Noetherian with $\dim(R[X]) = 1$. However, if R is Noetherian with $\dim(R) \geq 1$, then $\Omega(R) = \Omega(R[X]) = \mathbb{N}$ by (i) since $R[X]$ is Noetherian with $\dim(R[X]) \geq 2$. Example 5.8(e) shows that the converse of (h) is false. Also, in (h), for every positive integer n, there is a local Artinian ring R_n with $\Omega(R_n) = \{1, \ldots, n\}$; just let $R_n = \mathbb{Z}_{p^n}$ for p prime (cf. Example 5.8(a)).

We end this section with two questions. If $n \in \Omega(R)$ for some positive integer n, then is $m \in \Omega(R)$ for every integer m with $1 \leq m \leq n$? Is $\Omega(R_S) \subseteq \Omega(R)$ for S a multiplicatively closed subset of R?

6. STRONGLY *n*-ABSORBING IDEALS

In this final section, we introduce and study strongly n-absorbing ideals. It is well known that a proper ideal I of a ring R is a prime ideal of R if and only if whenever $I_1 I_2 \subseteq I$ for ideals I_1, I_2 of R, then either $I_1 \subseteq I$ or $I_2 \subseteq I$. Let n be a positive integer. We say that a proper ideal I of a ring R is a *strongly n-absorbing ideal* if whenever $I_1 \cdots I_{n+1} \subseteq I$ for ideals $I_1, \ldots, I_{n+1}$ of R, then the product of some n of the I_j's is in I. Thus a strongly 1-absorbing ideal is just a prime ideal, and the intersection of n prime ideals is a strongly n-absorbing ideal. It is clear that a strongly n-absorbing ideal of R is also an n-absorbing ideal of R, and in [3, Theorem 2.13], it was shown that these two concepts agree when $n = 2$. We conjecture that these two concepts agree for all positive integers n. In Corollary 6.9, we show that they agree for Prüfer domains.

Let I be a proper ideal of a ring R. If I is a strongly n-absorbing ideal of R for some positive integer n, we define $\omega_R^*(I) = \min\{n \mid I$ is a strongly n-absorbing ideal of $R\}$; otherwise, set $\omega_R^*(I) = \infty$ (we will just write $\omega^*(I)$ when the context is clear). Also, set $\omega_R^*(R) = 0$; so $\omega_R^*(I) \in \mathbb{N} \cup \{0, \infty\}$, $\omega_R^*(I) = 1$ if and only if I is a prime ideal of R, and $\omega_R(I) \leq \omega_R^*(I)$ for every ideal I of R. Define $\Omega^*(R) = \{\omega_R^*(I) \mid I$ is a proper ideal of $R\}$; so $\{1\} \subseteq \Omega(R) \subseteq \mathbb{N} \cup \{\infty\}$.

The interested reader may formulate results for ω^* and Ω^* analogous to those for ω and Ω. We will use the analog of Theorem 2.1(c) several times; namely, $\omega^*(I_1 \cap \cdots \cap I_m) \leq \omega^*(I_1) + \cdots + \omega^*(I_m)$ for ideals $I_1, \ldots, I_m$ of R. However, we next conjecture that $\omega_R = \omega_R^*$, and thus also $\Omega(R) = \Omega^*(R)$, for any ring R.

Conjecture 1. Let n be a positive integer. Then a proper ideal I of a ring R is a strongly n-absorbing ideal of R if and only I is an n-absorbing ideal of R (i.e., $\omega_R(I) = \omega_R^*(I)$ for every ideal I of R, and thus $\Omega(R) = \Omega^*(R)$).

Conjecture 2. Let n be a positive integer, and let I be an n-absorbing ideal of a ring R. Then $Rad(I)^n \subseteq I$.

We first show that Conjecture 1 implies Conjecture 2.

Theorem 6.1. *Let n be a positive integer and I a strongly n-absorbing ideal of a ring R. Then $Rad(I)^n \subseteq I$. In particular, Conjecture 1 implies Conjecture 2.*

Proof. Let $x_1, \ldots, x_n \in Rad(I)$, and let $J = (x_1, \ldots, x_n) \subseteq Rad(I)$. Then $x_i^n \in I$ for each $1 \leq k \leq n$ by Theorem 2.1(e), and thus $J^{n^n} \subseteq I$. Hence $J^n \subseteq I$ since I is a strongly n-absorbing ideal of R, and thus $Rad(I)^n \subseteq I$.

The "in particular" statement is clear. □

We next give some consequences of these two conjectures. The first theorem extends Theorem 2.14 and holds for n-absorbing ideals if Conjecture 1 holds.

Theorem 6.2. *Let n be a positive integer and I a strongly n-absorbing ideal of a ring R such that I has exactly $m(\leq n)$ minimal prime ideals $P_1, \ldots, P_m$. Then $P_1^{n_1} \cdots P_m^{n_m} \subseteq I$ for positive integers $n_1, \ldots, n_m$ with $n = n_1 + \cdots + n_m$. In particular, if $Rad(I) = P$ is a prime ideal of R, then $P^n \subseteq I$.*

Proof. Note that $m \leq n$ by Theorem 2.5. Let $J = Rad(I) = P_1 \cap \cdots \cap P_m$. Then $P_1 \cdots P_m \subseteq P_1 \cap \cdots \cap P_m = J$; so $(P_1 \cdots P_m)^n \subseteq J^n \subseteq I$ by Theorem 6.1, and thus $P_1^n \cdots P_m^n \subseteq I$. Since I is a strongly n-absorbing ideal of R, we have $P_1^{n_1} \cdots P_m^{n_m} \subseteq I$ for nonnegative integers $n_1, \ldots, n_m$ with $n = n_1 + \cdots + n_m$. Since $P_1^{n_1} \cdots P_m^{n_m} \subseteq I \subseteq P_i$ for each $1 \leq i \leq m$, we must have each $n_i \geq 1$.

The "in particular" statement is clear. □

Theorem 6.3. *Let P be a prime ideal of a ring R, n a positive integer, and suppose that Conjecture 2 holds.*

(a) *If P^n is a P-primary ideal of R and $P^n \subset P^{n-1}$, then $\omega(P^n) = n$.*
(b) *If P is a maximal ideal of R and $P^n \subset P^{n-1}$, then $\omega(P^n) = n$.*
(c) *Let I be a P-primary ideal of a ring R. If $P^n \subseteq I$ and $P^{n-1} \not\subseteq I$, then $\omega(I) = n$.*

Proof. (a) We have $\omega(P^n) \leq n$ by Theorem 3.1. If $\omega(P^n) \leq n-1$, then $P^{n-1} \subseteq P^n$ by Conjecture 2, a contradiction.

(b) If P is a maximal ideal of R, then P^n is P-primary.

(c) The proof is similar to that of (a). □

Remark 6.4. (a) Note that Theorem 6.3 improves the condition for $\omega(P^n) = n$ from $P^{n+1} \subset P^n$ to $P^n \subset P^{n-1}$ in the "moreover" statements of Lemma 2.8, Theorems 3.1, and 3.3.

(b) Let M be the maximal ideal of a quasilocal ring R with $\dim(R) = 0$ such that $M^{n+1} \subset M^n$ for every positive integer n. If Conjecture 2 holds, then $\omega_R(0) = \infty$. (If $\omega_R(0) = n < \infty$, then $M^n = 0$ by Conjecture 2, a contradiction.)

The next theorem gives a converse to Theorem 5.9. Note that if Conjecture 1 holds, then the hypothesis that 0 is a strongly n-absorbing ideal of R may be deleted.

Theorem 6.5. *Let n be a positive integer and R a ring such that* 0 *is a strongly n-absorbing ideal of R. Then every proper ideal of R is an n-absorbing ideal of R if and only if R is isomorphic to $R_1 \times \cdots \times R_m$, where $1 \leq m \leq n$, each R_i is a quasilocal ring with maximal ideal M_i, and there are positive integers $n_1, \ldots, n_m$ such that $n = n_1 + \cdots + n_m$ and $M_i^{n_i} = 0$ for each $1 \leq i \leq m$.*

Proof. Suppose that R is isomorphic to $T = R_1 \times \cdots \times R_m$, where $1 \leq m \leq n$, each R_i is a quasilocal ring with maximal ideal M_i, and there are positive integers $n_1, \ldots, n_m$ such that $n = n_1 + \cdots + n_m$ and $M_i^{n_i} = 0$ for each $1 \leq i \leq m$. First, observe that every proper ideal of each R_i is an M_i-primary ideal of R_i, and if I_i is a proper ideal of R_i, then $\omega_{R_i}(I_i) \leq n_i$ by Theorem 3.1 since $M_i^{n_i} = 0$. Let $I_1, \ldots, I_m$ be ideals of $R_1, \ldots, R_m$, respectively. Then $\omega_T(I_1 \times \cdots \times I_m) = \omega_{R_1}(I_1) + \cdots + \omega_{R_m}(I_m) \leq n_1 + \cdots + n_m = n$ by Corollary 4.8. Thus every proper ideal of T is an n-absorbing ideal of T, and hence the same holds for $R \cong T$ by Theorem 4.2(b).

Conversely, suppose that every proper ideal of R is an n-absorbing ideal of R. Then $\dim(R) = 0$ and R has $m \leq n$ maximal ideals by Theorem 5.9. Let $M_1, \ldots, M_m$ be the maximal ideals of R. Since 0 is a strongly n-absorbing ideal of R, we have $M_1^{n_1} \cdots M_m^{n_m} = 0$ for positive integers $n_1, \ldots, n_m$ with $n = n_1 + \cdots + n_m$ by Theorem 6.2. Thus R is isomorphic to $R/M_1^{n_1} \times \cdots \times R/M_m^{n_m}$ by the Chinese Remainder Theorem, and this product satisfies the desired properties. □

Our final theorem gives a case where the two concepts of n-absorbing and strongly n-absorbing ideals are equivalent. Note that the hypothesis in Theorem 6.6(1) that I is an n-absorbing ideal of R is redundant by Theorem 3.1. As corollaries, we have that the product of n maximal ideals is a strongly n-absorbing ideal (cf. Theorem 2.9), that every proper ideal of a Noetherian ring is a strongly n-absorbing ideal for some positive integer n (cf. Theorem 5.3), and that Conjecture 1 holds for the class of Prüfer domains (Corollary 6.9).

Theorem 6.6. *Let I be a P-primary ideal of a ring R and n a positive integer. Then the following statements are equivalent:*

(1) *I is an n-absorbing ideal of R and $P^n \subseteq I$;*
(2) *I is a strongly n-absorbing ideal of R.*

In particular, if P^n is P-primary, then P^n is a strongly n-absorbing ideal of R.

Proof. (1) $\Rightarrow$ (2) Suppose that $I_1 \cdots I_{n+1} \subseteq I$ for ideals $I_1, \ldots, I_{n+1}$ of R, but no product of n of the I_j's is contained in I. Then each I_j is contained in P since I is P-primary, and thus every product of n of the I_j's is contained in I because $P^n \subseteq I$. This is a contradiction; so there is a product of n of the I_j's that is contained in I.

(2) $\Rightarrow$ (1) This is clear by Theorem 6.2.

The "in particular" statement is clear by Theorem 3.1 and (1) $\Rightarrow$ (2) above. □

Corollary 6.7. *Let $M_1, \ldots, M_n$ be maximal ideals of a ring R. Then $I = M_1 \cdots M_n$ is a strongly n-absorbing ideal of R.*

Proof. The proof is essentially the same as the proof of Theorem 2.9, but with Theorem 6.6 replacing an appeal to Lemma 2.8 and using the analog of Theorem 2.1(c) for strongly absorbing ideals. □

Corollary 6.8. *Let R be a Noetherian ring. Then every proper ideal of R is a strongly n-absorbing ideal of R for some positive integer n.*

Proof. The proof is essentially the same as the proof of Theorem 5.3, but with Theorem 6.6 replacing an appeal to Theorem 3.1 and using the analog of Theorem 2.1(c) for strongly absorbing ideals. □

Corollary 6.9. *Let R be a Prüfer domain and n a positive integer. Then an ideal I of R is a strongly n-absorbing ideal of R if and only if I is an n-absorbing ideal of R. Moreover, $\omega(I) = \omega^*(I)$.*

Proof. We show that $\omega(I) = \omega^*(I)$ for I a nonzero, proper ideal of R with $\omega(I) = n$. By (the proof of) Theorem 5.7, we may assume that $I = P_1^{n_1} \cdots P_k^{n_k}$, where the P_i's are comaximal prime ideals of R, the n_i's are positive integers with $n_i = 1$ if P_i is idempotent, and $n = n_1 + \cdots + n_k$. Thus $\omega(I) \leq \omega^*(I) = \omega^*(P_1^{n_1} \cap \cdots \cap P_k^{n_k}) \leq \omega^*(P_1^{n_1}) + \cdots + \omega^*(P_k^{n_k}) \leq n_1 + \cdots + n_k = n = \omega(I)$ by the analog of Theorem 2.1(c) for strongly absorbing ideals and Theorem 6.6 (recall that each $P_i^{n_i}$ is a primary ideal of R by [8, Lemma 23.2(b)]). Hence $\omega(I) = \omega^*(I)$. □

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